On the Rational Dynamic Equation on Discrete Time Scales

$$x(\sigma(t)) = \frac{ax(t)+bx(\rho(t))}{c+d(x(t))^{m_1}(x(\rho(t)))^{m_2}}, \quad t \in \mathrm{T}$$

Sh. R. Elzeiny

Department of Mathematics, Faculty of Science, Al-Baha University, Kingdom of Saudi Arabia

Abstract**—** *In this paper, we study the global stability, periodicity character and some other properties of solutions of the rational dynamic equation on discrete time scales*

$$x(\sigma(t)) = \frac{ax(t)+bx(\rho(t))}{c+d(x(t))^{m_1}(x(\rho(t)))^{m_2}}, \quad t \in \mathrm{T},$$

where $a \geq 0, b, c, d \succ 0$ and $m_1 \geq 1,\ m_2 \geq 1.$

Keywords**— *Rational dynamic equation, Time scales, Equilibrium point, Global attractor, Periodicity, Boundedness, Invariant interval.***

I. INTRODUCTION

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [16], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice-once for differential equations and once again for difference equations.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus. Dynamic equations on a time scale have enormous potential for applications such as in population dynamics. For example, it can model insect population that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see Dosly et al. [10]).

Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [3] and the references cited therein. The first course on dynamic equations on time scales as in Bohner et al. [7]. For advance of dynamic equations on time scales as in Bohner et al. [6]. For completeness, we give a short introduction to the time scale calculus. the introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper.

Definition 1.1: A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. *Thus*

$$\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0,$$

i. e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are

$$[0,1]\cup[2,3],\ [0,1]\cup\mathbb{N},\ \textit{and the Cantor set},\ \text{while}$$

$$\mathbb{Q},\ \mathbb{R}\setminus\mathbb{Q},\ \mathbb{C}, (0,1),$$

i. e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales.

Throughout this paper, a time scale is denoted by the symbol T and has the topology that it inherits from the real numbers with the standard topology.

To reference points in the set T, the forward and backward jump operators are defined.

Definition 1.2: For $t \in T$, , the forward operator $\sigma : T \to T$ is defined by

$$\sigma(t) = \inf\{s \in T : s \succ t\},$$

and backward operator $\rho : T \to T$ is defined by

$$\rho(t) = \sup\{s \in T : s \prec t\}.$$

If T has a maximum t^*, and a minimum $t^\bullet$, then $\sigma(t^*) = t^*,\ and\ \rho(t^\bullet) = t^\bullet.$ When $\sigma(t) \neq t$ then t is called right scattered. When $\rho(t) \neq t$ then t is called left scattered.

Points t such that

$$\rho(t) \prec t \prec \sigma(t),\ \rho(t) \prec t = \sup T,\ or\ \sigma(t) \succ t = \inf T,$$

are called isolated points. If a time scale consists of only isolated points, then it is an isolated (discrete) time scale. Also, if $t \prec \sup T\ and\ \sigma(t) = t,\ then\, t$ is called right-dense, and if $t \succ \inf T\ and\ \rho(t) = t,\ then\, t$ is called left-dense. Points t that are either left-dense or right-dense are called dense.

Finally, the graininess operator $\mu : T \to [0, \infty)$ *is defined by* $\mu(t) = \sigma(t) - t$ and if $f : T \to \mathbb{R}$ is a function, then the function $f^\sigma : T \to \mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))\ \ for\ all\ t \in T,$$

i.e $f^\sigma = f \circ \sigma$ is the composition function of f with σ.

Definition 1.3: Assume $f : T \to \mathbb{R}$ is a function and let $t \in T$. If t is an isolated point, then we define

$$\lim_{s \to t} f(s) = f(t)$$

and we say f is continuous at t. When t is not isolated point, then when we write

$$\lim_{s \to t} f(s) = L,$$

it is understood that s approaches t in the time scale $(s \in T,\ s \neq t)$. We say f is continuous on $T.$ provided

$$\lim_{s \to t} f(s) = f(t)\ \ for\ all\ t \in T.$$

In particular, we have that any function defined on an isolated time scale (since all of its points are isolated points) is continuous. We say $f : [a,b]_T \to \mathbb{R}$ is continuous provided f is continuous at each point in $(a,b)_T$, f is left continuous at a, and f is right continuous at b.

In the time scale calculus, the functions are right-dense continuous which we now define.

Definition 1.4: A function $f:T\to\mathbb{R}$ is called right-dense continuous or briefly rd-continuous provided it is continuous at right-dense points in T. and its left-sided limits exist (finite) at left-dense points in T.. The set of rd-continuous functions $f:T\to\mathbb{R}$ will be denoted in this paper by

$$C_{rd}=C_{rd}(T)=C_{rd}(T,\mathbb{R})$$

In the next theorem we see that the jump operator is rd-continuous.

Theorem 1.1 (Bohner et al. [5]): The forward operator $\sigma:T\to T$ is increasing, rd-continuous, and

$$\sigma(t)\geq t \ \ for \ \ all \ \ t\in T,$$

and the jump operator is discontinuous at points which are left-dense and right-scattered.

Remark 1.1: The graininess function $\mu:T\to[0,\infty)$ is rd-continuous and μ is discontinuous at points in T that are both left-dense and right-scattered.

When $T=\mathbb{Z}$, the rational dynamic equation on discrete time scales

$$x(\sigma(t))=\frac{ax(t)+bx(\rho(t))}{c+d(x(t))^{m_1}(x(\rho(t)))^{m_2}},\ t\in T,\ \ where\ a\geq 0,\ b,c,d\succ 0\ and\ \ m_1,m_2\geq 1,$$

becomes the recursive sequence

$$x_{n+1}=\frac{ax_n+bx_{n-1}}{c+dx_n^{m_1}x_{n-1}^{m_2}},\ n=1,2,..., \tag{1.1}$$

$where\ a\geq 0,\ b,c,d\succ 0\ and\ \ m_1,m_2\geq 1.$

Now, the difference equations (as well as differential equations and delay differential equations) model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics, etc.[21]. The study of nonlinear difference equations is of paramount importance not only in their own right but in understanding the behavior of their differential counterparts.

There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms. There has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations [1, 2, 4, 8, 9, 11--15, 17-27].

This paper addresses, the global stability, periodicity character and boundedness of the solutions of the rational dynamic equation on discrete time scales

$$x(\sigma(t))=\frac{ax(t)+bx(\rho(t))}{c+d(x(t))^{m_1}(x(\rho(t)))^{m_2}},\ t\in T, \tag{1.2}$$

$where\ a\geq 0,\ b,c,d\succ 0\ and\ \ m_1,m_2\geq 1.$

When $T=\mathbb{Z}\ and\ \ m_1=m_2=1$, our equation reduces to equation which examined by Yang et al. [27].

Also, when $T=\mathbb{Z}\ and\ \ m_1=m_2=b=c=d=1\ and\ a=0$, our equation reduces to equation which examined by Cinar [8].

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let f be a continuous function defined on $I\times I$. Then, for initial conditions $x(\rho(t_0)), x(t_0)\in I$, it is easy to see that the dynamic equation on discrete time scales

$$x(\sigma(t))=f(x(t),x(\rho(t))),\ t\in T, \qquad (1.3)$$

has a unique solution $\{x(t):t\in T\}$, which is called a recursive sequence on time scales.

Definition 1.5: A point ω is called an equilibrium point of equation (1.3) if

$$\omega=f(\omega,\omega).$$

That is, $x(t)=\omega\ \ for\ t\in T$, is a solution of equation (1.3), or equivalently, ω is a fixed point of f.

Assume ω is an equilibrium point of equation (1.3) and $u=-f_{x(t)}(\omega,\omega),\ and\ v=--f_{x(\rho(t))}(\omega,\omega)$. Then the linearized equation associated with equation (1.3) about the equilibrium point ω is

$$z_{x(\sigma(t))}+uz_{x(t)}+vz_{x(\rho(t))}=0. \qquad (1.4)$$

The characteristic equation associated with equation (1.4) is

$$\lambda^2+u\lambda+v=0. \qquad (1.5)$$

Theorem 1.2 (Linearized stability theorem [18]):

$(1)\ If\ |u|\prec 1+v\ and\ v\prec 1,\ then\ \omega\ is\ locally\ asymptotically\ stable.$
$(2)\ If\ |u|\prec|1+v|\ and\ |v|\prec 1,\ then\ \omega\ is\ a\ repeller.$
$(3)\ If\ |u|\succ|1+v|\ and\ u^2\succ 4v,\ then\ \omega\ is\ a\ saddle\ po\text{int}.$
$(4)\ If\ |u|=|1+v|,\ then\ \omega\ is\ a\ non-hyperbolic\ po\text{int}.$

Definition 1.6: We say that a solution $\{x(t):t\in T\}$ of equation (1.3) is bounded if

$$|x(t)|\prec A\ \ for\ all\ t\in T.$$

Definition 1.7: (a) A solution $\{x(t):t\in T\}$ of equation (1.2) is said to be periodic with period υ if

$$x(t+\upsilon)=x(t)\ \ for\ all\ t\in T. \qquad (1.6)$$

(b) A solution $\{x(t):t\in T\}$ of equation (1.3) is said to be periodic with prime period υ, or υ-cycle if it is periodic with period υ and υ is the least positive integer for which (1.6) holds.

Definition 1.8: An interval $J\subseteq I$ is called invariant for equation (1.3) if every solution $\{x(t):t\in T\}$ of equation (1.3) with initial conditions $(x(\rho(t_0)),x(t_0))\in J\times J$ satisfies $x(t)\in J\ \ for\ all\ t\in T.$

For a real number $x(t_0)$ and a positive number R, let $O(x(t_0),R)=\{x(t):|x(t)-x(t_0)|\prec R\}$.

For other basic terminologies and results of difference equations the reader is referred to [18].

II MAIN RESULTS

2.1 Local asymptotic stability of the equilibrium points

Consider the rational dynamic equation on discrete time scales

$$x(\sigma(t))=\frac{ax(t)+bx(\rho(t))}{c+d(x(t))^{m_1}(x(\rho(t)))^{m_2}},\ t\in T, \tag{2.1}$$

where $a\ge 0,\ b,c,d\succ 0$ *and* $m_1,m_2\ge 1$.

Let $a'=\frac{a}{b},\ c'=\frac{c}{b},$ *and* $d'=\frac{d}{b}$. Then equation (2.1) can be rewritten as

$$x(\sigma(t))=\frac{a'x(t)+x(\rho(t))}{c'+d'(x(t))^{m_1}(x(\rho(t)))^{m_2}},\ t\in T. \tag{2.2}$$

The change of variables $y(t)=(d')^{1/m}x(t),\ m=m_1+m_2,$ followed by the change $y(t)=x(t)$ reduces the above equation to

$$x(\sigma(t))=\frac{px(t)+x(\rho(t))}{q+(x(t))^{m_1}(x(\rho(t)))^{m_2}},\ t\in T, \tag{2.3}$$

where $p=a'=\frac{a}{b},$ *and* $q=c'=\frac{c}{b}$. Hereafter, we focus our attention on equation (2.3) instead of equation (2.1). Assume that,

$$m_1=\frac{k_1}{2n_1+1},\ and\ \ m_2=\frac{k_2}{2n_2+1},\ k_1,k_2\in\mathbb{Z}_+,\ and\ n_1,n_2\in\mathbb{Z}_+\cup\{0\}.$$

When m is the ratio of odd positive integers, equation (2.3) has only two equilibrium points

$$\alpha=0,\ and\ \beta=\sqrt[m]{p+1-q}.$$

When m is positive rational number and numerator's even positive integers, equation (2.3) has unique equilibrium point

$$\alpha=0\ if\ \ q\ge p+1,$$

and three equilibrium points:

$$\alpha=0,\ \ \beta=\sqrt[m]{p+1-q},\ \ and\ \gamma=-\sqrt[m]{p+1-q}\ \ \ if\ \ p+1\succ q.$$

Furthermore, suppose that, at least one of m_1 *and* $m_2 = \dfrac{2k_3+1}{2n_3},\ k_3, n_3 \in \mathbb{Z}_+$.

Then, in this case, we consider $x(t) \geq 0$ *for all* $t \in T$.

When $q \geq p+1,$ equation (2.3) has unique equilibrium point

$$\alpha = 0.$$

When $p+1 \succ q,$ however, equation (2.3) has the following two equilibrium points:

$$\alpha = 0,\ and\ \beta = \sqrt[m]{p+1-q}.$$

The local asymptotic behavior of $\alpha = 0$ is characterized by the following result.

THEOREM 2.1:

(1) If $q \succ p+1$, then α is locally asymptotically stable.

(2) If $q \prec 1-p$, then α is a repeller.

(3) If $|q-1| \prec p$, then α is a saddle point.

Proof: The Linearized equation associated with equation (2.3) about the equilibrium point $\alpha = 0$ is

$$z(\sigma(t)) - \frac{p}{q} z(t) - \frac{1}{q} z(\rho(t)) = 0,\ t \in T. \qquad (2.4)$$

Now, when $T = \mathbb{Z}$, then equation (2.4) becomes

$$z_{n+1} - \frac{p}{q} z_n - \frac{1}{q} z_{n-1} = 0,\ n \in \mathbb{Z}. \qquad (2.5)$$

The characteristic equation associated with equation (2.5) is

$$\lambda^2 - \frac{p}{q}\lambda - \frac{1}{q} = 0,\ where\ z(t) = \lambda^t, \qquad (2.6)$$

when $T = r^{\mathbb{N}},\ r \succ 1,$ then equation (2.4) becomes

$$z_{r^{n+1}} - \frac{p}{q} z_{r^n} - \frac{1}{q} z_{r^{n-1}} = 0,\ n \in \mathbb{N}. \qquad (2.7)$$

The characteristic equation associated with equation (2.7) is

$$\lambda^2 - \frac{p}{q}\lambda - \frac{1}{q} = 0,\ where\ z(t) = \lambda^{\log_r(t)}, \qquad (2.8)$$

when $T = h\mathbb{Z},$ then equation (2.4) becomes

$$z_{h(n+1)} - \frac{p}{q} z_{hn} - \frac{1}{q} z_{h(n-1)} = 0,\ n \in \mathbb{Z}. \qquad (2.9)$$

The characteristic equation associated with equation (2.9) is

$$\lambda^2 - \frac{p}{q}\lambda - \frac{1}{q} = 0,\ where\ z(t) = \lambda^{t/h}, \tag{2.10}$$

and when $T = \mathbb{N}_0^2$, then equation (2.4) becomes

$$z_{(n+1)^2} - \frac{p}{q} z_{n^2} - \frac{1}{q} z_{(n-1)^2} = 0,\ n \in \mathbb{N}_0. \tag{2.11}$$

The characteristic equation associated with equation (2.9) is

$$\lambda^2 - \frac{p}{q}\lambda - \frac{1}{q} = 0,\ where\ z(t) = \lambda^{\sqrt{t}}. \tag{2.12}$$

Hence, the characteristic equation associated with equation (2.4) is

$$\lambda^2 - \frac{p}{q}\lambda - \frac{1}{q} = 0,\ for\ all\ t \in T. \tag{2.13}$$

$Let\ u = -\frac{p}{q},\ and\ v = -\frac{1}{q}.$

(1) The result follows from Theorem 1.2 (1) and the following relations

$$|u| - (1+v) = \frac{p}{q} - (1 - \frac{1}{q}) = \frac{(p+1)-q}{q} \prec 0,\quad and\quad v = -\frac{1}{q} \prec 0 \prec 1.$$

(2) The result follows from Theorem 1.2(2) and the following relations

$$|u| - |1+v| = \frac{p}{q} - |1 - \frac{1}{q}| = \frac{p}{q} - |\frac{q-1}{q}| = \frac{p}{q} - \frac{1-q}{q} = \frac{q-(1-p)}{q} \prec 0,$$

and

$$|v| = \frac{1}{q} \succ 1.$$

(3) The result follows from Theorem 1.2 (3) and the following relations

$$|u| - |1+v| = \frac{p}{q} - |1 - \frac{1}{q}| = \frac{p}{q} - |\frac{q-1}{q}| = \frac{p - |q-1|}{q} \succ 0,$$

and|

$$u^2 - 4v = (\frac{p}{q})^2 + \frac{4}{q} \succ 0.$$

Now, the local asymptotic behavior of β and γ are characterized by the following result.

that heorem 2.2: Assume that

$$p+1 \succ q,\ m_1 \prec \frac{p}{p+1-q},\ and\ m_2 \prec \frac{p+2}{p+1-q},$$

then both β *and* γ are locally asymptotically stable.

Proof : The Linearized equation associated with equation (2.3) about the equilibrium β is

$$z(\sigma(t)) + \frac{m_1(p+1-q)-p}{p+1} z(t) + \frac{m_2(p+1-q)-1}{p+1} z(\rho(t)) = 0,\ t \in T. \qquad (2.14)$$

Hence, the characteristic equation associated with equation (2.14) is

$$\lambda^2 + \frac{m_1(p+1-q)-p}{p+1}\lambda + \frac{m_2(p+1-q)-1}{p+1} = 0, \quad for\ all\ t \in T.$$

$$Let\ u = \frac{m_1(p+1-q)-p}{p+1} \quad and \quad v = \frac{m_2(p+1-q)-1}{p+1}$$

Then, from Theorem 1.2 (1) and the following relations

$$|u| - (v+1) = \left|\frac{m_1(p+1-q)-p}{p+1}\right| - \left(\frac{m_2(p+1-q)-1}{p+1}+1\right) = \frac{p-m_1(p+1-q)}{p+1} - \left(\frac{m_2(p+1-q)+p}{p+1}\right)$$
$$= \frac{-m(p+1-q)}{p+1} \prec 0,$$

and

$$v = \frac{m_2(p+1-q)-1}{p+1} \prec 1,$$

we conclude that β is locally asymptotically stable. Similarly, we can prove that γ is locally asymptotically stable.

2.2 Boundedness of solutions of equation (2.3)

In this section, we study the boundedness of solution of equation (2.3).

Theorem 2.3: Suppose that $p+1 \le q$ *and* m_i is positive rational number and numeraor's even positive integers, $i = 1, 2$. Then the solution of equation (2.3) is bounded for all $t \in T$.

Proof: We argue that $|x(t)| \prec A\ for\ all\ t \in T$ by induction on t.

Case(1): If $T=\mathbb{Z}$. Given any initial conditions $|x_{-1}|\prec A$ *and* $|x_0|\prec A$, we argue that $|x_n|\prec A$ *for all* $n\in\mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2$ *and* $n-1(n\geq 1).$ That is,

$$|x_{n-2}|\prec A \text{ and } |x_{n-1}|\prec A.$$

Now, we consider x_n, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_n|\leq \frac{1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|}(p|x_{n-1}|+|x_{n-2}|)\leq \frac{(p+1)A}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|} \qquad (2.15)$$

Since,

$$|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|=q+x_{n-1}^{m_1}x_{n-2}^{m_2}\geq q\succ 0,$$

$$\frac{1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|}\leq \frac{1}{q}. \qquad (2.16)$$

From (2.15) and (2.16), we obtain,

$$|x_n|\prec \frac{(p+1)A}{q}\prec A \text{ for all } n.$$

Case(2): If $T=h\mathbb{Z}=\{hk:h\succ 0 \text{ and } k\in\mathbb{Z}\}.$ Given any initial conditions $|x_{-h}|\prec A$ *and* $|x_0|\prec A$, we argue that $|x_{hn}|\prec A$ *for all* $n\in\mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2$ *and* $n-1(n\geq 1).$ That is,

$$|x_{h(n-2)}|\prec A \text{ and } |x_{h(n-1)}|\prec A.$$

Now, we consider x_{hn}, where we put hn instead of $h(n+1)$ in equation (2.3),

$$|x_{hn}|\leq \frac{1}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|}(p|x_{h(n-1)}|+|x_{h(n-2)}|)\prec \frac{(p+1)A}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|}. \qquad (2.17)$$

Since,

$$|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|=q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}\geq q\succ 0,$$

$$\frac{1}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|}\leq \frac{1}{q}. \qquad (2.18)$$

From (2.17) and (2.18), we obtain,

$$|x_{hn}| \prec \frac{(p+1)A}{q} \prec A \ for \ all \ n.$$

Case (3): If $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$ Given any initial conditions $|x_1| \prec A \ and \ |x_0| \prec A,$ we argue that $|x_k| \prec A \ for\,all \ k = n^2, \ n \in \mathbb{N}_0$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 0,1.$ Suppose the assertion is true for $(n-2)^2 \ and \ (n-1)^2.$ That is,

$$|x_{(n-2)^2}| \prec A \ and \ |x_{(n-1)^2}| \prec A.$$

Now, we consider x_{n^2}, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_{n^2}| \le \frac{1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}(p|x_{(n-1)^2}| + |x_{(n-2)^2}|) \prec \frac{(p+1)A}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}. \quad (2.19)$$

Since,

$$|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}| = q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2} \ge q \succ 0,$$

$$\frac{1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|} \le \frac{1}{q}. \quad (2.20)$$

From (2.19) and (2.20), we obtain,

$$|x_k| \prec \frac{(p+1)A}{q} \prec A \ for \ all \ k.$$

Case(4): If $T = r^{\mathbb{N}}, \ r \succ 1.$ Given any initial conditions $|x_{1/r}| \prec A \ and \ |x_1| \prec A,$ we argue that $|x_k| \prec A \ for\,all \ k \in r^n, \ n \in \mathbb{N}$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 1/r, 1.$ Suppose the assertion is true for $k = r^{n-2} \ and \ k = r^{n-1}, \ where \ n \in \mathbb{N}.$ That is,

$$|x_{r^{n-2}}| \prec A \ and \ |x_{r^{n-1}}| \prec A.$$

Now, we consider x_{r^2}, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_{r^n}| \le \frac{1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}(p|x_{r^{n-1}}| + |x_{r^{n-2}}|) \prec \frac{(p+1)A}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}. \quad (2.21)$$

Since,

$$|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}| = q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2} \ge q,$$

$$\frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|} \le \frac{1}{q}. \quad (2.22)$$

From (2.21) and (2.22), we obtain,

$$|x_k| \prec \frac{(p+1)A}{q} \prec A \ \ for \ all \ k.$$

This completes the proof.

(2.3) Periodic solutions of equation

In this section, we study the existence of periodic solutions of equation (2.3).

The following theorem states the necessary and sufficient conditions that this equation has a periodic solutions.

Theorem 2.4: Equation (2.3) has prime period two solutions if and only if

$$p+q \succ 1 \ and \ 0 \prec p \neq 1, \ where \ m_1 = m_2 = \text{the ratio of odd positive integers.}$$

Proof: First, suppose that, there exists a prime period two solution

$$\ldots, \varphi, \psi, \varphi, \psi, \ldots,$$

of equation (2.3). We will prove that $p+q \succ 1.$ We see from equation (2.3) that

$$\varphi = \frac{p\psi + \varphi}{q+(\varphi\psi)^k} \ and \ \psi = \frac{p\varphi + \psi}{q+(\varphi\psi)^k}, \ \ k = m_1 = m_2.$$

Then,

$$\varphi - \psi = \frac{p(\psi-\varphi)+(\varphi-\psi)}{q+(\varphi\psi)^k}.$$

Since, $\varphi - \psi \neq 0, \ \ 1 = \dfrac{1-p}{q+(\varphi\psi)^k}.$

Therefore,

$$\varphi\psi = \sqrt[k]{1-p-q}, \tag{2.23}$$

and so,

$$\varphi + \psi = \frac{(p+1)(\varphi+\psi)}{q+(\varphi\psi)^k}.$$

Hence,

$$(\varphi+\psi)[q-p-1+(\varphi\psi)^k] = 0. \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$(\varphi+\psi)[q-p-1+1-p-q] = 0. \ Then, \ \ (\varphi+\psi)(-2p) = 0.$$

Since, $p \succ 0,$

$$(\varphi+\psi)=0. \qquad (2.25)$$

It is clear now, from equation (2.23) and (2.25) that φ *and* ψ are the two distinct roots of the quadratic equation

$$t^2+\sqrt[k]{1-p-q}=0 \ or \ t^2+\sqrt[k]{p+q-1}=0,$$

So, $p+q-1 \succ 0.$ *Then,* $p+q \succ 1.$

Second, suppose that $p+q \succ 1$. We will show that equation (2.3) has a prime period two solutions. Assume that

$$\varphi=\sqrt[k]{p+q-1} \ \ and \ \ \psi=-\sqrt[k]{p+q-1}.$$

Therefore φ *and* ψ are distinct real numbers. Set,

$$x(\rho(t_0))=\varphi \ and \ x(t_0)=\psi, \ t_0\in T.$$

We wish to show that

$$x(\sigma(t_0))=x(\rho(t_0))=\varphi.$$

It follows from equation (2.3) that

$$x(\sigma(t_0))=\frac{px(t_0)+x(\rho(t_0))}{q+(x(t_0)x(\rho(t_0))^k}=\frac{(1-p)\sqrt[k]{p+q-1}}{q-(p+q-1)}=\frac{(1-p)\sqrt[k]{p+q-1}}{1-p}.$$

Since, $p\neq 1$, $x(\sigma(t_0))=\sqrt[k]{p+q-1}=\varphi.$

Similarly as before one can easily show that

$$x(\sigma(t))=\varphi, \ where \ x(\rho(t))=\varphi \ and \ x(t)=\psi.$$

Thus equation (2.3) has the prime period two solution

$$\ldots,\varphi,\psi,\varphi,\psi,\ldots,$$

where φ *and* ψ are the distinct roots of the quadratic equation (2.3). The proof is completed.

To examine the global attractivity of the equilibrium points of equation (2.3), we first need to determine the invariant intervals for equation (2.3).

2.4 Invariant intervals and global attractivity of the zero equilibria

In this subsection, we determine the family of invariant intervals centered at $\alpha=0.$

Theorem 2.5: Assume $q \succ p+1$. Then for any positive real number

$$A\leq(q-(p+1))^{1/m},$$

The interval $O(0,A)=(-A,A)$ is invariant for equation (2.3).

Proof: We consider $T=\mathbb{Z},\ h\mathbb{Z}=\{hk:h\succ 0\ and\ k\in\mathbb{Z}\},\ \mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\ and\ r^{\mathbb{N}},\ r\succ 1.$

Case (1): If $T=\mathbb{Z}$. Given any initial conditions $|x_{-1}|\prec A\ and\ |x_0|\prec A,$ we argue that $|x_n|\prec A\ for\,all\ n\in\mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2\ and\ n-1(n\geq 1).$ That is,

$$|x_{n-2}|\prec A\leq (q-(p+1))^{1/m}\ and\ |x_{n-1}|\prec A\leq (q-(p+1))^{1/m}.$$

Now, we consider x_n, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q+x_{n-1}^{m_1}x_{n-2}^{m_2}\succ q-A^m\geq q-(q-(p+1))=p+1\succ 0,$$

$$\frac{p+1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|}\leq 1. \qquad (2.26)$$

Thus,

$$\begin{aligned}|x_n|&\leq\frac{1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|}(p|x_{n-1}|+|x_{n-2}|)\\&\leq(\frac{p+1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|})\max\{|x_{n-1}|,|x_{n-2}|\}\\&\prec\max\{|x_{n-1}|,|x_{n-2}|\}\prec A\ \ for\,all\ n\in\mathbb{Z}.\end{aligned}$$

Case (2): If $T=h\mathbb{Z}=\{hk:h\succ 0\ and\ k\in\mathbb{Z}\}.$ Given any initial conditions $|x_{-h}|\prec A\ and\ |x_0|\prec A,$ we argue that $|x_{hn}|\prec A\ for\,all\ n\in\mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-h,0.$ Suppose the assertion is true for $n-2\ and\ n-1(n\geq 1).$ That is,

$$|x_{h(n-2)}|\prec A\leq (q-(p+1))^{1/m}\ and\ |x_{h(n-1)}|\prec A\leq (q-(p+1))^{1/m}.$$

Now, we consider x_{hn}, where we put hn instead of $h(n+1)$ in equation (2.3). Since,

$$q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}\succ q-A^m\geq q-(q-(p+1))=p+1\succ 0,$$

$$\frac{p+1}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|}\leq 1. \qquad (2.27)$$

Thus,

$$|x_{hn}| \le \frac{1}{|q + x_{h(n-1)}^{m_1} x_{h(n-2)}^{m_2}|}(p|x_{h(n-1)}| + |x_{h(n-2)}|)$$
$$\le (\frac{p+1}{|q + x_{h(n-1)}^{m_1} x_{h(n-2)}^{m_2}|}) \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\}$$
$$\prec \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\} \prec A \ \ for\,all \ \ n \in \mathbb{Z}.$$

Case (3): If $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$ Given any initial conditions $|x_1| \prec A \ and \ |x_0| \prec A,$ we argue that $|x_k| \prec A \ \ for\,all \ \ k = n^2, \ n \in \mathbb{N}_0$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 0,1.$ Suppose the assertion is true for $(n-2)^2 \ \ and \ \ (n-1)^2.$ That is,

$$|x_{(n-2)^2}| \prec A \le (q-(p+1))^{1/m} \ \ and \ |x_{(n-1)^2}| \prec A \le (q-(p+1))^{1/m}.$$

Now, we consider x_{n^2}, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2} \succ q - A^m \ge q - (q-(p+1)) = p+1 \succ 0,$$

$$\frac{p+1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|} \le 1. \qquad (2.28)$$

Thus,

$$|x_{n^2}| \le \frac{1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}(p|x_{(n-1)^2}| + |x_{(n-2)^2}|)$$
$$\le (\frac{p+1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}) \max\{|x_{(n-1)^2}|, |x_{(n-2)^2}|\}$$
$$\prec \max\{|x_{(n-1)^2}|, |x_{(n-2)^2}|\} \prec A \ \ for\,all \ \ k = n^2.$$

Case (4): If $T = r^{\mathbb{N}}, \ r \succ 1.$ Given any initial conditions $|x_{1/r}| \prec A \ and \ |x_1| \prec A,$ we argue that $|x_k| \prec A \ \ for\,all \ \ k \in r^n, \ n \in \mathbb{N}$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 1/r, 1.$ Suppose the assertion is true for $k = r^{n-2} \ \ and \ \ k = r^{n-1}, \ where \ n \in \mathbb{N}.$ That is,

$$|x_{r^{n-2}}| \prec A \le (q-(p+1))^{1/m} \ \ and \ |x_{r^{n-1}}| \prec A \le (q-(p+1))^{1/m}.$$

Now, we consider x_{r^2}, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2} \succ q - A^m \ge q - (q-(p+1)) = p+1 \succ 0,$$

$$\frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|} \le 1. \qquad (2.29)$$

Thus,

$$|x_{r^n}| \le \frac{1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}(p|x_{r^{n-1}}| + |x_{r^{n-2}}|)$$
$$\le (\frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}) \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\}$$
$$\prec \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\} \prec A \ \textit{for all } n.$$

This completes the proof.

Now, we investigate the global attractivity of the equilibrium point $\alpha = 0.$

Lemma 2.1: Assume that $T = \mathbb{Z},\ q \succ p+1,\ and\ m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that

$$R = (q-(p+1))^{1lm}, \qquad (2.30)$$

and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_n\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q - (\max\{|x_{-1}|,|x_0|\})^m}. \qquad (2.31)$$

Then,

$$R_1 \in (0,1)\ and\, |x_n| \le R_1^{n/2} \times \max\{|x_{-1}|,|x_0|\},\ n = 1,2,... \quad . \qquad (2.32)$$

Proof: From Theorem 2.5, we have

$$|x_n| \prec A \le R,\ n = -1,0,1,2,... \quad ,$$

where we put n instead of $(n+1)$ in equation (2.3). Then,

$$(\max\{|x_{-1}|,|x_0|\})^m \prec R^m.$$

Thus,

$$q - (\max\{|x_{-1}|,|x_0|\})^m \succ q - R^m = 1+p \succ 0.$$

Hence,

$$0 \prec \frac{1+p}{q - (\max\{|x_{-1}|,|x_0|\})^m} \prec 1. \ \textit{This means that } R_1 \in (0,1).$$

Now, we prove that (2.32), by induction on n . From equation (2.3), we have

$$|x_n| \le \frac{1}{|q + x_{n-1}^{m_1} x_{n-2}^{m_2}|}(p|x_{n-1}| + |x_{n-2}|)$$

$$\le (\frac{p+1}{|q + x_{n-1}^{m_1} x_{n-2}^{m_2}|}) \max\{|x_{n-1}|, |x_{n-2}|\} \qquad (2.33)$$

$$\prec \max\{|x_{n-1}|, |x_{n-2}|\}, \ \textit{from} \ (2.26). \qquad (2.34)$$

Then, from (2.33), we obtain

$$|x_1| \le (\frac{p+1}{|q + x_0^{m_1} x_{-1}^{m_2}|}) \max\{|x_0|, |x_{-1}|\}. \qquad (2.35)$$

But

$$|q + x_0^{m_1} x_{-1}^{m_2}| \ge q - |x_0^{m_1}||x_{-1}^{m_2}| \ge q - (\max\{|x_0|, |x_{-1}|\})^m \succ q - R^m = 1 + p \succ 0.$$

Then,

$$|q + x_0^{m_1} x_{-1}^{m_2}| \ge q - (\max\{|x_0|, |x_{-1}|\})^m \succ 0.$$

From (2.35), we have

$$|x_1| \le (\frac{p+1}{|q + x_0^{m_1} x_{-1}^{m_2}|}) \max\{|x_0|, |x_{-1}|\} \le \frac{p+1}{q - (\max\{|x_0|, |x_{-1}|\})^m} \max\{|x_0|, |x_{-1}|\}$$
$$\le R_1 \times \max\{|x_0|, |x_{-1}|\} \prec R_1^{1/2} \times \max\{|x_0|, |x_{-1}|\},$$

And from (2.33) and (2.34), we obtain

$$|x_2| \le (\frac{p+1}{|q + x_1^{m_1} x_0^{m_2}|}) \max\{|x_1|, |x_0|\} \le \frac{p+1}{q - (\max\{|x_1|, |x_0|\})^m} \max\{|x_1|, |x_0|\}$$
$$\prec \frac{p+1}{q - (\max\{\max\{|x_0|, |x_{-1}|\}, |x_0|\})^m} \times \max\{\max\{|x_0|, |x_{-1}|\}, |x_0|\}$$
$$\prec \frac{p+1}{q - (\max\{|x_{-1}|, |x_0|\})^m} \times \max\{|x_{-1}|, |x_0|\} = R_1 \times \max\{|x_{-1}|, |x_0|\}.$$

Thus, the inequality (2.32) holds for $n = 1, 2.$ Suppose that the inequality (2.32) holds for $n-1$ and $n-2\,(n \ge 3)$, respectively. By (2.34), we have

$$|x_n| \le \max\{|x_0|, |x_{-1}|\} \ \textit{for all} \ n.$$

So,

$$0 \prec \frac{p+1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|} \le \frac{p+1}{q-|x_{n-1}|^{m_1}|x_{n-2}|^{m_2}} \le \frac{p+1}{q-(\max\{|x_0|,|x_{-1}|\})^m} = R_1.$$

Then, from (2.33), we have

$$\begin{aligned}|x_n| &\le \left(\frac{p+1}{|q+x_{n-1}^{m_1}x_{n-2}^{m_2}|}\right)\max\{|x_{n-1}|,|x_{n-2}|\} \le R_1 \max\{|x_{n-1}|,|x_{n-2}|\}\\ &\le R_1 \max\{R_1^{(n-1)/2}\max\{|x_0|,|x_{-1}|\}, R_1^{(n-2)/2}\max\{|x_0|,|x_{-1}|\}\}\\ &\le R_1^{n/2}\max\{R_1^{1/2}\max\{|x_0|,|x_{-1}|\},\max\{|x_0|,|x_{-1}|\}\}\\ &\le R_1^{n/2}\max\{|x_0|,|x_{-1}|\}.\end{aligned}$$

This completes the inductive proof of (2.32).

Lemma 2.2: Assume that $T = h\mathbb{Z} = \{hk : h \in (0,1) \ and \ k \in \mathbb{Z}\},\ q \succ p+1,\ and\ m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_{kn} : h \in (0,1) \ and \ n \in \mathbb{Z}\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q-(\max\{|x_{-h}|,|x_0|\})^m}. \qquad (2.36)$$

Then,

$$R_1 \in (0,1) \ and |x_{hn}| \le R_1^{hn/2} \times \max\{|x_{-h}|,|x_0|\},\ n = 1,2,... \quad . \qquad (2.37)$$

Proof: From Theorem 2.5, we have

$$|x_{hn}| \prec A \le R,\ n = -1,0,1,... \quad ,$$

Where we put hn instead of $h(n+1)$ in equation (2.3). Then,

$$(\max\{|x_{-h}|,|x_0|\})^m \prec R^m.$$

Thus,

$$q-(\max\{|x_{-h}|,|x_0|\})^m \succ q - R^m = 1+p \succ 0.$$

Hence,

$$0 \prec \frac{1+p}{q-(\max\{|x_{-h}|,|x_0|\})^m} \prec 1. \ This\ means\ that\ R_1 \in (0,1).$$

Now, we prove that (2.37), by induction on n . From equation (2.3), we have

$$|x_{hn}| \leq \frac{1}{|q + x_{h(n-1)}^{m_1} x_{h(n-2)}^{m_2}|}(p|x_{h(n-1)}| + |x_{h(n-2)}|)$$

$$\leq (\frac{p+1}{|q + x_{h(n-1)}^{m_1} x_{h(n-2)}^{m_2}|}) \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\} \quad (2.38)$$

$$\prec \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\}, \ from \ (2.27). \quad (2.39)$$

Then, from (2.38), we obtain

$$|x_h| \leq (\frac{p+1}{|q + x_0^{m_1} x_{-h}^{m_2}|}) \max\{|x_0|, |x_{-h}|\}. \quad (2.40)$$

But

$$|q + x_0^{m_1} x_{-h}^{m_2}| \geq q - |x_0^{m_1}||x_{-h}^{m_2}| \geq q - (\max\{|x_0|, |x_{-h}|\})^m \succ q - R^m = 1 + p \succ 0.$$

Then,

$$|q + x_0^{m_1} x_{-h}^{m_2}| \geq q - (\max\{|x_0|, |x_{-h}|\})^m \succ 0.$$

From (2.40), we have

$$|x_h| \leq (\frac{p+1}{|q + x_0^{m_1} x_{-h}^{m_2}|}) \max\{|x_0|, |x_{-h}|\} \leq \frac{p+1}{q - (\max\{|x_0|, |x_{-h}|\})^m} \max\{|x_0|, |x_{-h}|\}$$
$$\leq R_1 \times \max\{|x_0|, |x_{-h}|\} \prec R_1^{h/2} \times \max\{|x_0|, |x_{-h}|\},$$

And from (2.38) and (2.39), we obtain

$$|x_{2h}| \leq (\frac{p+1}{|q + x_h^{m_1} x_0^{m_2}|}) \max\{|x_h|, |x_0|\} \leq \frac{p+1}{q - (\max\{|x_h|, |x_0|\})^m} \max\{|x_h|, |x_0|\}$$
$$\prec \frac{p+1}{q - (\max\{\max\{|x_0|, |x_{-h}|\}, |x_0|\})^m} \times \max\{\max\{|x_0|, |x_{-h}|\}, |x_0|\}$$
$$\prec \frac{p+1}{q - (\max\{|x_{-h}|, |x_0|\})^m} \times \max\{|x_{-h}|, |x_0|\} = R_1 \times \max\{|x_{-h}|, |x_0|\}$$
$$\prec R_1^{2h/2} \times \max\{|x_{-h}|, |x_0|\}.$$

Thus, the inequality (2.37) holds for $n = 1, 2.$ Suppose that the inequality (2.37) holds for $h(n-1)$ and $h(n-2)\,(n \geq 3)$, respectively. By (2.39), we have

$$|x_n| \leq \max\{|x_0|, |x_{-h}|\} \ for \ all \ n.$$

So,

$$0 \prec \frac{p+1}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|} \le \frac{p+1}{q-|x_{h(n-1)}|^{m_1}|x_{h(n-2)}|^{m_2}} \le \frac{p+1}{q-(\max\{|x_0|,|x_{-h}|\})^m} = R_1.$$

Then, from (2.38), we have

$$\begin{aligned}|x_{hn}| &\le (\frac{p+1}{|q+x_{h(n-1)}^{m_1}x_{h(n-2)}^{m_2}|})\max\{|x_{h(n-1)}|,|x_{h(n-2)}|\}\\ &\le R_1\max\{|x_{h(n-1)}|,|x_{h(n-2)}|\} \le R_1^h\max\{|x_{h(n-1)}|,|x_{h(n-2)}|\}\\ &\le R_1^h\max\{R_1^{h(n-1)/2}\max\{|x_0|,|x_{-h}|\},R_1^{h(n-2)/2}\max\{|x_0|,|x_{-h}|\}\}\\ &\le R_1^{hn/2}\max\{R_1^{h/2}\max\{|x_0|,|x_{-h}|\},\max\{|x_0|,|x_{-h}|\}\}\\ &\le R_1^{hn/2}\max\{|x_0|,|x_{-h}|\}.\end{aligned}$$

This completes the inductive proof of (2.37).

Lemma 2.3: Assume that $T=\mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\ q\succ p+1,\ and\ m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f:O(0,R)\times O(0,R)\to O(0,R).$$

Let $\{x_k:k=n^2\ and\ n\in\mathbb{N}_0\}$ be a solution of this equation and

$$R_1=\frac{1+p}{q-(\max\{|x_1|,|x_0|\})^m}. \tag{2.41}$$

Then,

$$R_1\in(0,1)\ and\,|x_k|\le R_1^{k/2}\times\max\{|x_{-1}|,|x_0|\},\ k=1,4,\ldots\quad. \tag{2.42}$$

Proof: From Theorem 2.5, we have

$$|x_k|\prec A\le R,\ k=1,2,\ldots\quad,$$

where we put n instead of $(n+1)$ in equation (2.3). Then,

$$(\max\{|x_1|,|x_0|\})^m\prec R^m.$$

Thus,

$$q-(\max\{|x_1|,|x_0|\})^m\succ q-R^m=1+p\succ 0.$$

Hence,

$$0\prec\frac{1+p}{q-(\max\{|x_1|,|x_0|\})^m}\prec 1.\ This\ means\ that\ R_1\in(0,1).$$

Now, we prove that (2.42), by induction on k . From equation (2.3), we have

$$|x_{n^2}| \le \frac{1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}(p|x_{(n-1)^2}| + |x_{(n-2)^2}|)$$

$$\le (\frac{p+1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}) \max\{|x_{(n-1)^2}|, |x_{(n-2)^2}|\} \quad (2.43)$$

$$\prec \max\{|x_{(n-1)^2}|, |x_{(n-2)^2}|\}, \; from \; (2.28). \quad (2.44)$$

Then, from (2.43), we obtain

$$|x_1| \le (\frac{p+1}{|q + x_0^{m_1} x_1^{m_2}|}) \max\{|x_0|, |x_1|\}. \quad (2.45)$$

But

$$|q + x_0^{m_1} x_1^{m_2}| \ge q - |x_0^{m_1}||x_1^{m_2}| \ge q - (\max\{|x_0|, |x_1|\})^m \succ q - R^m = 1 + p \succ 0.$$

Then,

$$|q + x_0^{m_1} x_1^{m_2}| \ge q - (\max\{|x_0|, |x_1|\})^m \succ 0.$$

From (2.45), we have

$$|x_1| \le (\frac{p+1}{|q + x_0^{m_1} x_1^{m_2}|}) \max\{|x_0|, |x_1|\} \le \frac{p+1}{q - (\max\{|x_0|, |x_1|\})^m} \max\{|x_0|, |x_1|\}$$

$$\le R_1 \times \max\{|x_0|, |x_1|\} \prec R_1^{1/2} \times \max\{|x_0|, |x_1|\},$$

And from (2.43) and (2.44), we obtain

$$|x_4| \le (\frac{p+1}{|q + x_1^{m_1} x_0^{m_2}|}) \max\{|x_1|, |x_0|\} \le \frac{p+1}{q - (\max\{|x_1|, |x_0|\})^m} \max\{|x_1|, |x_0|\}$$

$$\prec \frac{p+1}{q - (\max\{\max\{|x_0|, |x_1|\}, |x_0|\})^m} \times \max\{R_1 \max\{|x_0|, |x_1|\}, |x_0|\}$$

$$\prec \frac{p+1}{q - (\max\{|x_1|, |x_0|\})^m} \times R_1 \times \max\{|x_1|, |x_0|\} \prec R_1^2 \times \max\{|x_1|, |x_0|\}$$

$$\prec R_1^{4/2} \times \max\{|x_1|, |x_0|\}.$$

Thus, the inequality (2.42) holds for $n = 1, 2.$ Suppose that the inequality (2.42) holds for $(n-1)^2$ and $(n-2)^2$, respectively. By (2.44), we have

$$|x_{n^2}| \le \max\{|x_{(n-1)^2}|, |x_{(n-2)^2}|\} \; for\, all \; n \in \mathbb{N}_0.$$

So,

$$0 \prec \frac{p+1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|} \le \frac{p+1}{q - |x_{(n-1)^2}|^{m_1} |x_{(n-2)^2}|^{m_2}} \le \frac{p+1}{q - (\max\{|x_0|,|x_1|\})^m} = R_1.$$

Then, from (2.43), we have

$$\begin{aligned} |x_{n^2}| &\le \left(\frac{p+1}{|q + x_{(n-1)^2}^{m_1} x_{(n-2)^2}^{m_2}|}\right) \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \le R_1 \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \\ &\le \max\{R_1^{(n-1)^2/2} \max\{|x_0|,|x_1|\}, R_1^{(n-2)^2/2} \max\{|x_0|,|x_1|\}\} \\ &\le R_1^{(n-2)^2/2} \max\{|x_0|,|x_1|\},\ n = 3,4,... \\ &\le R_1^{n^2/2} \max\{|x_0|,|x_1|\},\ n = 1,2,... \quad . \end{aligned}$$

This completes the inductive proof of (2.42).

Lemma 2.4: Assume that $T = r^{\mathbb{N}}, 1 \prec r \le 2, q \succ p+1,$ *and* m is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_k : k = r^n,\ n \in \mathbb{N} \text{ and } 1 \prec r \le 2\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q - (\max\{|x_{1/r}|,|x_1|\})^m}. \tag{2.46}$$

Then,

$$R_1 \in (0,1) \text{ and } |x_k| \le R_1^{k/2} \times \max\{|x_{1/r}|,|x_1|\}. \tag{2.47}$$

Proof: From Theorem 2.5, we have

$$|x_k| \prec A \le R,\ k = r^n,\ n \in \mathbb{N}, \text{ and } 1 \prec r \le 2,$$

where we put n instead of $(n+1)$ in equation (2.3). Then,

$$(\max\{|x_{1/r}|,|x_1|\})^m \prec R^m.$$

Thus,

$$q - (\max\{|x_{1/r}|,|x_1|\})^m \succ q - R^m = 1 + p \succ 0.$$

Hence,

$$0 \prec \frac{1+p}{q - (\max\{|x_{1/r}|,|x_1|\})^m} \prec 1. \text{ This means that } R_1 \in (0,1).$$

Now, we prove that (2.47), by induction on n. From equation (2.3), we have

$$|x_{r^n}| \leq \frac{1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}(p|x_{r^{n-1}}| + |x_{r^{n-2}}|)$$

$$\leq (\frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}) \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\} \qquad (2.48)$$

$$\prec \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\}, \ from \ (2.29). \qquad (2.49)$$

Then, from (2.48), we obtain

$$|x_r| \leq (\frac{p+1}{|q + x_1^{m_1} x_{1/r}^{m_2}|}) \max\{|x_{1/r}|, |x_1|\}. \qquad (2.50)$$

But

$$|q + x_1^{m_1} x_{1/r}^{m_2}| \geq q - |x_1^{m_1}||x_{1/r}^{m_2}| \geq q - (\max\{|x_{1/r}|, |x_1|\})^m \succ q - R^m = 1 + p \succ 0.$$

Then,

$$|q + x_{1/r}^{m_1} x_1^{m_2}| \geq q - (\max\{|x_{1/r}|, |x_1|\})^m \succ 0.$$

From (2.50), we have

$$|x_r| \leq (\frac{p+1}{|q + x_{1/r}^{m_1} x_1^{m_2}|}) \max\{|x_{1/r}|, |x_1|\} \leq \frac{p+1}{q - (\max\{|x_{1/r}|, |x_1|\})^m} \max\{|x_{1/r}|, |x_1|\}$$
$$\leq R_1 \times \max\{|x_{1/r}|, |x_1|\} \prec R_1^{r/2} \times \max\{|x_{1/r}|, |x_1|\},$$

And from (2.48) and (2.49), we obtain

$$|x_{r^2}| \leq (\frac{p+1}{|q + x_r^{m_1} x_1^{m_2}|}) \max\{|x_r|, |x_1|\} \leq \frac{p+1}{q - (\max\{|x_r|, |x_1|\})^m} \max\{|x_r|, |x_1|\}$$
$$\prec \frac{p+1}{q - (\max\{\max\{|x_{1/r}|, |x_1|\}, |x_1|\})^m} \times \max\{R_1^{r/2} \max\{|x_{1/r}|, |x_1|\}, |x_1|\}$$
$$\prec \frac{p+1}{q - (\max\{|x_{1/r}|, |x_1|\})^m} \times R_1^{r/2} \times \max\{|x_{1/r}|, |x_1|\}$$
$$\prec R_1 \times R_1^{r/2} \times \max\{|x_{1/r}|, |x_1|\} \prec R_1^{r/2} \times R_1^{r/2} \times \max\{|x_{1/r}|, |x_1|\}$$
$$\prec R_1^{2r/2} \times \max\{|x_{1/r}|, |x_1|\} \leq R_1^{r^2/2} \times \max\{|x_{1/r}|, |x_1|\}.$$

Thus, the inequality (2.47) holds for $k = r, r^2$. Suppose that the inequality (2.47) holds for $k = r^{n-2}$ *and* $k = r^{n-1}, n \in \mathbb{N},$ respectively. By (2.49), we have

$$|x_{r^n}| \leq \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\} \ for\, all \ n.$$

So,

$$0 \prec \frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|} \le \frac{p+1}{q - |x_{r^{n-1}}|^{m_1} |x_{r^{n-2}}|^{m_2}} \le \frac{p+1}{q - (\max\{|x_{1/r}|, |x_1|\})^m} = R_1.$$

Then, from (2.48), we have

$$\begin{aligned} |x_{r^n}| &\le \left(\frac{p+1}{|q + x_{r^{n-1}}^{m_1} x_{r^{n-2}}^{m_2}|}\right) \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\} \le R_1 \max\{|x_{r^{n-1}}|, |x_{r^{n-2}}|\} \\ &\le \max\{R_1^{r^{n-1}/2} \max\{|x_{1/r}|, |x_1|\}, R_1^{r^{n-2}/2} \max\{|x_{1/r}|, |x_1|\}\} \\ &\le R_1^{r^{n-2}/2} \times \max\{|x_{1/r}|, |x_1|\}\}, \; n = 3, 4, \ldots \\ &\le R_1^{r^{n}/2} \times \max\{|x_{1/r}|, |x_1|\}, \; n = 1, 2, 3, \ldots \quad . \end{aligned}$$

This completes the inductive proof of (2.47).

Theorem 2.6: Assume that $q \succ p+1, \; and \; m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds, and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Then the equilibrium point $\alpha = 0$ of the equation (2.3) is a global attractor.

Proof: From Lemmas $2.i, \; i = 1,2,3,4, \; x(t) \to \alpha = 0 \; when \; t \to \infty,$ then the equilibrium point $\alpha = 0$ of the equation (2.3) is a global attractor.

Next, we determine the family of invariant intervals centered at $\beta = \sqrt[m]{(p+1) - q} \;\; when \;\; p+1 \succ q.$

2.5 Invariant intervals and global attractivity of the nonzero equilibria

In this subsection, we consider the discrete time scales

$T = \mathbb{Z}, \; h\mathbb{Z} = \{hk : h \succ 0 \; and \; k \in \mathbb{Z}\}, \;\; \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}, \qquad and \;\; r^{\mathbb{N}}, \; r \succ 1, \; m_1 = m_2 = 1, \; and \; (p+1) \succ q.$

This means that, we determine the family of invariant intervals centered at $\beta = \sqrt{(p+1) - q} \;\; when \;\; p+1 \succ q.$ To this end, we establish the following relation.

$$\begin{aligned} x(\sigma(t)) - \beta &= \frac{px(t) + x(\rho(t))}{q + x(t)x(\rho(t))} - \frac{p\beta + \beta}{q + \beta^2} \\ &= \left(\frac{px(t) + x(\rho(t))}{q + x(t)x(\rho(t))} - \frac{p\beta + x(\rho(t))}{q + \beta x(\rho(t))}\right) + \left(\frac{p\beta + x(\rho(t))}{q + \beta x(\rho(t))} - \frac{p\beta + \beta}{q + \beta^2}\right) \\ &= \frac{pq - x^2(\rho(t))}{(q + x(t)x(\rho(t)))(q + \beta x(\rho(t)))}(x(t) - \beta) \\ &\quad + \frac{q - p\beta^2}{(q + \beta x(\rho(t)))(q + \beta^2)}(x(\rho(t)) - \beta). \end{aligned}$$

In view of $\beta^2 = p + 1 - q,$ the above equality is reduced to

$$x(\sigma(t))-\beta=\frac{pq-x^2(\rho(t))}{(q+x(t)x(\rho(t)))(q+\beta x(\rho(t)))}(x(t)-\beta)$$
$$+\frac{q-p}{q+\beta x(\rho(t))}(x(\rho(t))-\beta).$$

Thus,

$$\begin{aligned}|x(\sigma(t))-\beta|&\le\left|\frac{pq-x^2(\rho(t))}{(q+x(t)x(\rho(t)))(q+\beta x(\rho(t)))}\right||x(t)-\beta|\\&+\left|\frac{q-p}{q+\beta x(\rho(t))}\right||x(\rho(t))-\beta|\\&\le\left(\left|\frac{pq-x^2(\rho(t))}{(q+x(t)x(\rho(t)))(q+\beta x(\rho(t)))}\right|+\left|\frac{q-p}{q+\beta x(\rho(t))}\right|\right)\\&\times\max\{|x(\rho(t))-\beta|,|x(t)-\beta|\}.\end{aligned}\tag{2.51}$$

Consider the function

$$g(x,y)=\frac{pq-y^2}{(q+xy)(q+\beta y)}+\frac{|p-q|}{(q+\beta y)},\ x\succ 0,\ y\in(0,\sqrt{pq}).\tag{2.52}$$

The following result is obvious.

Lemma 2.5: The function g is strictly decreasing in x and decreasing in y.

Consider the function

$$h(x)=g(x,x)=\frac{pq-x^2}{(q+x^2)(q+\beta x)}+\frac{|p-q|}{(q+\beta x)},\ x\in(0,\sqrt{pq}).$$

We have the following result.

Lemma 2.6: Assume that $q\ge p.$ Then, $h(x)\prec 1\ \ for\ \ x\in(0,\sqrt{pq}).$

Proof: Clearly, h is strictly decreasing in x. Since $q\ge p,$ then

$$h(x)=\frac{pq-x^2}{(q+x^2)(q+\beta x)}+\frac{p-q}{(q+\beta x)}.$$

The result follows from $h(0)=1.$

Now, we are ready to describe the family of nested invariant intervals centered at β.

Theorem 2.7: Assume that $\max\{1,p\}\prec q\prec p+1.$

i- If $q \geq \frac{4(p+1)}{4+p}$, then for every positive number $A \leq \beta,$ the interval

$$O(\beta, A) = (\beta - A, \beta + A)$$

is invariant for equation (2.3).

ii- If $q \prec \frac{4(p+1)}{4+p}$, then for every positive number $A \leq \sqrt{pq} - \beta,$ the interval

$$O(\beta, A) = (\beta - A, \beta + A)$$

is invariant for equation (2.3).

Proof: First, we note that

$$pq - \beta^2 = pq - [(p+1) - q] = (1+p)(q-1) \succ 0.$$

That is, β between $0 \; and \; \sqrt{pq}.$ Now assume

$$A \leq \min\{\beta, \sqrt{pq} - \beta\} = \{ \begin{matrix} \beta & if \; q \geq \frac{4(p+1)}{4+p}, \\ \sqrt{pq} - \beta & if \; q \prec \frac{4(p+1)}{4+p}. \end{matrix}$$

Case (1): If $T = \mathbb{Z}$, given any initial conditions

$$| x_{-1} - \beta | \prec A \; and \; | x_0 - \beta | \prec A,$$

we argue that

$$| x_n - \beta | \prec A \;\; for \;\; all \;\; n \;\; by \, induction \, on \;\; n.$$

The proof is similar to the proof of Theorem 4.4 in [27] and will be omitted.

Case 2: If $T = h\mathbb{Z}$, given any initial conditions

$$| x_{-h} - \beta | \prec A \; and \; | x_0 - \beta | \prec A,$$

we argue that

$$| x_{hn} - \beta | \prec A \;\; for \;\; all \;\; n \;\; by \, induction \, on \;\; n.$$

It follows from the given initial assertion is true for $n = -1, 0.$

Suppose the asseration is true for $h(n-2) \; and \; h(n-1).$. That is,

$$| x_{h(n-2)} - \beta | \prec A \; and \; | x_{h(n-1)} - \beta | \prec A.$$

Now, we consider x_{hn} where we put hn instead of $h(n+1)$ in equation (2.51), Lemma 2.5 and Lemma 2.6, we have

$$
\begin{aligned}
|x_{hn}-\beta| &\le \left(\left|\frac{pq-x_{h(n-2)}^2}{(q+x_{h(n-1)}x_{h(n-2)})(q+\beta x_{h(n-2)})}\right|+\left|\frac{q-p}{q+\beta x_{h(n-2)}}\right|\right)\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\
&\le \left(\frac{pq-x_{h(n-2)}^2}{(q+x_{h(n-1)}x_{h(n-2)})(q+\beta x_{h(n-2)})}+\frac{q-p}{q+\beta x_{h(n-2)}}\right)\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\
&\le g(x_{h(n-1)},x_{h(n-2)})\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\
&\le g(\min\{x_{h(n-1)},x_{h(n-2)}\},\min\{x_{h(n-1)},x_{h(n-2)}\})\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\
&\le h(\min\{x_{h(n-1)},x_{h(n-2)}\})\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\
&\le \max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\}\prec A.
\end{aligned}
$$

This completes the inductive proof.

When $T=\mathbb{N}_0^2$ *and* $r^{\mathbb{N}}$, $r\succ 1$, the proof will be omitted.

By an analogous argument to that for Theorem 2.7, we can obtain the following result about the family of nested invariant intervals centered at γ .

Theorem 2.8: Assume that $\max\{1,p\}\prec q\prec p+1.$

i- If $q\ge\frac{4(p+1)}{4+p}$, then for every positive number $A\le|\gamma|$, the interval

$$O(\gamma,A)=(\gamma-A,\gamma+A)$$

is invariant for equation (2.3).

ii- If $q\prec\frac{4(p+1)}{4+p}$, then for every positive number $A\le\sqrt{pq}-|\gamma|$, the interval

$$O(\gamma,A)=(\gamma-A,\gamma+A)$$

is invariant for equation (2.3).

Now, we investigate the global attractivity of the equilibrium point β *and* γ.

Lemma 2.8: Assume that $T=\mathbb{Z}$, *and* $\max\{1,p\}\prec q\prec p+1.$ Suppose that

$$R=\min\{\beta,\sqrt{pq}-\beta\}, \qquad (2.53)$$

and consider equation (2.3) with the restriction that

$$f:O(\beta,R)\times O(\beta,R)\to O(\beta,R).$$

Let $\{x_n\}$ be a solution of this equation and

$$M=\begin{cases} \frac{pq-x_0^2}{(q+x_0^2)(q+\beta x_0)}+\frac{q-p}{q+\beta x_0} & if\ |x_{-1}-\beta|\le|x_0-\beta|\ \&\ x_0\le\beta, \\ \frac{pq-x_0^2}{(q+(2\beta-x_0)^2)(q+\beta(2\beta-x_0))}+\frac{q-p}{q+\beta(2\beta-x_0)} & if\ |x_{-1}-\beta|\le|x_0-\beta|\ \&\ x_0\ge\beta, \\ \frac{pq-x_{-1}^2}{(q+x_{-1}^2)(q+\beta x_{-1})}+\frac{q-p}{q+\beta x_{-1}} & if\ |x_{-1}-\beta|\ge|x_0-\beta|\ \&\ x_{-1}\le\beta, \\ \frac{pq-x_{-1}^2}{(q+(2\beta-x_{-1})^2)(q+\beta(2\beta-x_{-1}))}+\frac{q-p}{q+\beta(2\beta-x_{-1})} & if\ |x_{-1}-\beta|\ge|x_0-\beta|\ \&\ x_{-1}\ge\beta. \end{cases} \quad (2.54)$$

Then,

$$M\in(0,1)\ and\ |x_n-\beta|\le M^{n/2}\times\max\{|x_{-1}-\beta|,|x_0-\beta|\},\ n=1,2,... \quad . \quad (2.55)$$

The proof is similar to the proof of Lemma 6.1 in [27] and will be omitted.

Lemma 2.9: Assume that $T=h\mathbb{Z}=\{hk:h\in(0,1)\ and\ k\in\mathbb{Z}\},\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that (2.53) holds and consider equation (2.3) with the restriction that

$$f:O(\beta,R)\times O(\beta,R)\to O(\beta,R).$$

Let $\{x_{hn}:h\in(0,1)\ and\ n\in\mathbb{Z}\}$ be a solution of this equation and

$$M=\begin{cases} \frac{pq-x_0^2}{(q+x_0^2)(q+\beta x_0)}+\frac{q-p}{q+\beta x_0} & if\ |x_{-h}-\beta|\le|x_0-\beta|\ \&\ x_0\le\beta, \\ \frac{pq-x_0^2}{(q+(2\beta-x_0)^2)(q+\beta(2\beta-x_0))}+\frac{q-p}{q+\beta(2\beta-x_0)} & if\ |x_{-h}-\beta|\le|x_0-\beta|\ \&\ x_0\ge\beta, \\ \frac{pq-x_{-h}^2}{(q+x_{-h}^2)(q+\beta x_{-h})}+\frac{q-p}{q+\beta x_{-h}} & if\ |x_{-h}-\beta|\ge|x_0-\beta|\ \&\ x_{-h}\le\beta, \\ \frac{pq-x_{-h}^2}{(q+(2\beta-x_{-h})^2)(q+\beta(2\beta-x_{-h}))}+\frac{q-p}{q+\beta(2\beta-x_{-h})} & if\ |x_{-h}-\beta|\ge|x_0-\beta|\ \&\ x_{-h}\ge\beta. \end{cases} \quad (2.56)$$

Then,

$$M\in(0,1)\ and\ |x_{hn}-\beta|\le M^{hn/2}\times\max\{|x_{-h}-\beta|,|x_0-\beta|\},\ n=1,2,... \quad . \quad (2.57)$$

Proof: The result that $M\in(0,1)$ follows from Lemma 2.6. Next, we prove equation (2.57) by induction on n. First, set

$$A=\begin{cases} \beta-x_0 & if\ |x_{-h}-\beta|\le|x_0-\beta|\ \&\ x_0\le\beta, \\ x_0-\beta & if\ |x_{-h}-\beta|\le|x_0-\beta|\ \&\ x_0\ge\beta, \\ \beta-x_{-h} & if\ |x_{-h}-\beta|\ge|x_0-\beta|\ \&\ x_{-h}\le\beta, \\ x_{-h}-\beta & if\ |x_{-h}-\beta|\le|x_0-\beta|\ \&\ x_{-h}\ge\beta. \end{cases}$$

Then, $A \le \beta.$ By Lemma 2.5 and Theorem 2.7, we derive that for all n ,

$$\frac{pq - x_{h(n-2)}^2}{(q + x_{h(n-1)} x_{h(n-2)})(q + \beta x_{h(n-2)})} + \frac{q - p}{q + \beta x_{h(n-2)}} \le M,$$

where we put hn instead of $h(n+1)$. By equation (2.51), we have

$$\begin{aligned} |x_h - \beta| &\le \left(\left|\frac{pq - x_{-h}^2}{(q + x_0 x_{h(n-2)})(q + \beta x_{-h})}\right| + \left|\frac{q - p}{q + \beta x_{-h}}\right|\right) \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\} \\ &\le M \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\} \\ &\le M^{h/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}, \ and \end{aligned}$$

$$\begin{aligned} |x_{2h} - \beta| &\le \left(\left|\frac{pq - x_0^2}{(q + x_0 x_h)(q + \beta x_0)}\right| + \left|\frac{q - p}{q + \beta x_0}\right|\right) \times \max\{|x_0 - \beta|, |x_h - \beta|\} \\ &\le M \times \max\{|x_0 - \beta|, M \times \max\{|x_0 - \beta|, |x_h - \beta|\}\} \\ &\le M \times \max\{|x_0 - \beta|, |x_h - \beta|\} \\ &\le M^{2h/2} \times \max\{|x_0 - \beta|, |x_h - \beta|\}. \end{aligned}$$

So equation (2.57) holds for $n = 1, 2$. Suppose that equation (2.57) holds for $h(n-1)$ and $h(n$-2), respectively. By the inductive hypothesis, we have

$$\begin{aligned} |x_{hn} - \beta| &\le \left(\frac{pq - x_{h(n-2)}^2}{(q + x_{h(n-1)} x_{h(n-2)})(q + \beta x_{h(n-2)})} + \frac{q - p}{q + \beta x_{h(n-2)}}\right) \times \max\{|x_{h(n-1)} - \beta|, |x_{h(n-2)} - \beta|\} \\ &\le M \max\{M^{h(n-1)/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}, M^{h(n-2)/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}\} \\ &\le M^h \max\{M^{h(n-1)/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}, M^{h(n-2)/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}\} \\ &\le M^{nh/2} \max\{M^{h/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}, \max\{|x_0 - \beta|, |x_{-h} - \beta|\}\} \\ &\le M^{nh/2} \max\{|x_0 - \beta|, |x_{-h} - \beta|\}. \end{aligned}$$

This completes the inductive proof of equation (2.57).

Lemma 2.10: Assume that $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\},\ and\ \max\{1, p\} \prec q \prec p + 1.$ Suppose that (2.53) holds and consider equation (2.3) with the restriction that

$$f : O(\beta, R) \times O(\beta, R) \to O(\beta, R).$$

Let $\{x_k : k = n^2 \ and\ n \in \mathbb{N}_0\}$ be a solution of this equation and

$$M=\{\begin{array}{ll} \frac{pq-x_0^2}{(q+x_0^2)(q+\beta x_0)}+\frac{q-p}{q+\beta x_0} & if\ |x_1-\beta|\le|x_0-\beta|\,\&\,x_0\le\beta, \\ \frac{pq-x_0^2}{(q+(2\beta-x_0)^2)(q+\beta(2\beta-x_0))}+\frac{q-p}{q+\beta(2\beta-x_0)} & if\ |x_1-\beta|\le|x_0-\beta|\,\&\,x_0\ge\beta, \\ \frac{pq-x_1^2}{(q+x_1^2)(q+\beta x_1)}+\frac{q-p}{q+\beta x_1} & if\ |x_1-\beta|\ge|x_0-\beta|\,\&\,x_1\le\beta, \\ \frac{pq-x_1^2}{(q+(2\beta-x_1)^2)(q+\beta(2\beta-x_1))}+\frac{q-p}{q+\beta(2\beta-x_1)} & if\ |x_1-\beta|\ge|x_0-\beta|\,\&\,x_1\ge\beta. \end{array} \quad (2.58)$$

Then,

$$M\in(0,1)\ and\,|x_k-\beta|\le M^{k/2}\times\max\{|x_1-\beta|,|x_0-\beta|\},\ k=1,4,... \quad . \quad (2.59)$$

Lemma 2.11: Assume that $T=r^{\mathbb{N}},\ 1\prec r\le 2,\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that (2.53) holds and consider equation (2.3) with the restriction that

$$f:O(\beta,R)\times O(\beta,R)\to O(\beta,R).$$

Let $\{x_k : k=r^n,\ 1\prec r\le 2,\ and\ n\in\mathbb{N}\}$ be a solution of this equation and

$$M=\{\begin{array}{ll} \frac{pq-x_1^2}{(q+x_1^2)(q+\beta x_1)}+\frac{q-p}{q+\beta x_1} & if\ |x_{1/r}-\beta|\le|x_1-\beta|\,\&\,x_1\le\beta, \\ \frac{pq-x_1^2}{(q+(2\beta-x_1)^2)(q+\beta(2\beta-x_1))}+\frac{q-p}{q+\beta(2\beta-x_1)} & if\ |x_{1/r}-\beta|\le|x_1-\beta|\,\&\,x_1\ge\beta, \\ \frac{pq-x_{1/r}^2}{(q+x_{1/r}^2)(q+\beta x_{1/r})}+\frac{q-p}{q+\beta x_{1/r}} & if\ |x_{1/r}-\beta|\ge|x_1-\beta|\,\&\,x_{1/r}\le\beta, \\ \frac{pq-x_{1/r}^2}{(q+(2\beta-x_{1/r})^2)(q+\beta(2\beta-x_{1/r}))}+\frac{q-p}{q+\beta(2\beta-x_{1/r})} & if\ |x_{1/r}-\beta|\ge|x_1-\beta|\,\&\,x_{1/r}\ge\beta. \end{array} \quad (2.60)$$

Then,

$$M\in(0,1)\ and\,|x_k-\beta|\le M^{k/2}\times\max\{|x_{1/r}-\beta|,|x_1-\beta|\}. \quad (2.61)$$

Theorem 2.9: Assume that $\max\{1,p\}\prec q\prec p+1,$ and (2.53) holds. Consider equation (2.3) with the restriction that

$$f:O(\beta,R)\times O(\beta,R)\to O(\beta,R).$$

Then the equilibrium point β of the equation (2.3) is a global attractor.

Proof: From Lemmas $2.i,\ i=8,9,10,11,$ *we obtain* $x(t)\to\beta$ *when* $t\to\infty,$ then the equilibrium point β of the equation (2.3) is a global attractor.

At the end of this section , we can establish the following results related to the global attractivity of γ .

Lemma 2.12: Assume that $T=\mathbb{Z},\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that

$$R=\min\{|\gamma|,\sqrt{pq}-|\gamma|\}, \qquad (2.62)$$

and consider equation (2.3) with the restriction that

$$f:O(\gamma,R)\times O(\gamma,R)\to O(\gamma,R).$$

Let $\{x_n\}$ be a solution of this equation and

$$N=\begin{cases}\dfrac{pq-x_0^2}{(q+x_0^2)(q+\gamma x_0)}+\dfrac{q-p}{q+\gamma x_0} & if\ |x_{-1}-\gamma|\le|x_0-\gamma|\,\&\,x_0\le\gamma,\\ \dfrac{pq-x_0^2}{(q+(2\gamma-x_0)^2)(q+\gamma(2\gamma-x_0))}+\dfrac{q-p}{q+\gamma(2\gamma-x_0)} & if\ |x_{-1}-\gamma|\le|x_0-\gamma|\,\&\,x_0\ge\gamma,\\ \dfrac{pq-x_{-1}^2}{(q+x_{-1}^2)(q+\gamma x_{-1})}+\dfrac{q-p}{q+\gamma x_{-1}} & if\ |x_{-1}-\gamma|\ge|x_0-\gamma|\,\&\,x_{-1}\le\gamma,\\ \dfrac{pq-x_{-1}^2}{(q+(2\gamma-x_{-1})^2)(q+\gamma(2\gamma-x_{-1}))}+\dfrac{q-p}{q+\gamma(2\gamma-x_{-1})} & if\ |x_{-1}-\gamma|\ge|x_0-\gamma|\,\&\,x_{-1}\ge\gamma.\end{cases} \qquad (2.63)$$

Then,

$$N\in(0,1)\ and\,|x_n-\beta|\le N^{n/2}\times\max\{|x_{-1}-\gamma|,|x_0-\gamma|\},\ n=1,2,\ldots\quad . \qquad (2.64)$$

Lemma 2.13: Assume that $T=h\mathbb{Z}=\{hk:h\in(0,1)\ and\ k\in\mathbb{Z}\},\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that (2.62) holds and consider equation (2.3) with the restriction that

$$f:O(\gamma,R)\times O(\gamma,R)\to O(\gamma,R).$$

Let $\{x_{hn}:h\in(0,1)\ and\ n\in\mathbb{Z}\}$ be a solution of this equation and

$$N=\begin{cases}\dfrac{pq-x_0^2}{(q+x_0^2)(q+\gamma x_0)}+\dfrac{q-p}{q+\gamma x_0} & if\ |x_{-h}-\gamma|\le|x_0-\gamma|\,\&\,x_0\le\gamma,\\ \dfrac{pq-x_0^2}{(q+(2\gamma-x_0)^2)(q+\gamma(2\gamma-x_0))}+\dfrac{q-p}{q+\gamma(2\gamma-x_0)} & if\ |x_{-h}-\gamma|\le|x_0-\gamma|\,\&\,x_0\ge\gamma,\\ \dfrac{pq-x_{-h}^2}{(q+x_{-h}^2)(q+\gamma x_{-h})}+\dfrac{q-p}{q+\gamma x_{-h}} & if\ |x_{-h}-\gamma|\ge|x_0-\gamma|\,\&\,x_{-h}\le\gamma,\\ \dfrac{pq-x_{-h}^2}{(q+(2\gamma-x_{-h})^2)(q+\gamma(2\gamma-x_{-h}))}+\dfrac{q-p}{q+\gamma(2\gamma-x_{-h})} & if\ |x_{-h}-\gamma|\ge|x_0-\gamma|\,\&\,x_{-h}\ge\gamma.\end{cases} \qquad (2.65)$$

Then,

$$N\in(0,1)\ and\,|x_{hn}-\gamma|\le N^{hn/2}\times\max\{|x_{-h}-\gamma|,|x_0-\gamma|\},\ n=1,2,\ldots\quad . \qquad (2.66)$$

Lemma 2.14: Assume that $T=\mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that (2.62) holds and consider equation (2.3) with the restriction that

$$f:O(\gamma,R)\times O(\gamma,R)\to O(\gamma,R).$$

Let $\{x_k:k=n^2\ and\ n\in\mathbb{N}_0\}$ be a solution of this equation and

$$N=\{\begin{array}{ll}\frac{pq-x_0^2}{(q+x_0^2)(q+\gamma x_0)}+\frac{q-p}{q+\gamma x_0} & if\ |x_1-\gamma|\le|x_0-\gamma|\ \&\ x_0\le\gamma,\\ \frac{pq-x_0^2}{(q+(2\gamma-x_0)^2)(q+\gamma(2\gamma-x_0))}+\frac{q-p}{q+\gamma(2\gamma-x_0)} & if\ |x_1-\gamma|\le|x_0-\gamma|\ \&\ x_0\ge\gamma,\\ \frac{pq-x_1^2}{(q+x_1^2)(q+\gamma x_1)}+\frac{q-p}{q+\gamma x_1} & if\ |x_1-\gamma|\ge|x_0-\gamma|\ \&\ x_1\le\gamma,\\ \frac{pq-x_1^2}{(q+(2\gamma-x_1)^2)(q+\gamma(2\gamma-x_1))}+\frac{q-p}{q+\gamma(2\gamma-x_1)} & if\ |x_1-\gamma|\ge|x_0-\gamma|\ \&\ x_1\ge\gamma.\end{array}\qquad(2.67)$$

Then,

$$N\in(0,1)\ and\,|x_k-\gamma|\le N^{k/2}\times\max\{|x_1-\gamma|,|x_0-\gamma|\},\ k=1,4,\ldots\quad.\qquad(2.68)$$

Lemma 2.15: Assume that $T=r^{\mathbb{N}},\ 1\prec r\le 2,\ and\ \max\{1,p\}\prec q\prec p+1.$ Suppose that (2.62) holds and consider equation (2.3) with the restriction that

$$f:O(\gamma,R)\times O(\gamma,R)\to O(\gamma,R).$$

Let $\{x_k:k=r^n,\ 1\prec r\le 2,\ and\ n\in\mathbb{N}\}$ be a solution of this equation and

$$N=\{\begin{array}{ll}\frac{pq-x_1^2}{(q+x_1^2)(q+\gamma x_1)}+\frac{q-p}{q+\gamma x_1} & if\ |x_{1/r}-\gamma|\le|x_1-\gamma|\ \&\ x_1\le\gamma,\\ \frac{pq-x_1^2}{(q+(2\gamma-x_1)^2)(q+\gamma(2\gamma-x_1))}+\frac{q-p}{q+\gamma(2\gamma-x_1)} & if\ |x_{1/r}-\gamma|\le|x_1-\gamma|\ \&\ x_1\ge\gamma,\\ \frac{pq-x_{1/r}^2}{(q+x_{1/r}^2)(q+\gamma x_{1/r})}+\frac{q-p}{q+\gamma x_{1/r}} & if\ |x_{1/r}-\gamma|\ge|x_1-\gamma|\ \&\ x_{1/r}\le\gamma,\\ \frac{pq-x_{1/r}^2}{(q+(2\gamma-x_{1/r})^2)(q+\gamma(2\gamma-x_{1/r}))}+\frac{q-p}{q+\gamma(2\gamma-x_{1/r})} & if\ |x_{1/r}-\gamma|\ge|x_1-\gamma|\ \&\ x_{1/r}\ge\gamma.\end{array}\qquad(2.69)$$

Then,

$$N\in(0,1)\ and\,|x_k-\gamma|\le N^{k/2}\times\max\{|x_{1/r}-\gamma|,|x_1-\gamma|\}.\qquad(2.70)$$

Theorem 2.10: Assume that $\max\{1,p\}\prec q\prec p+1,$ and (2.62) holds. Consider equation (2.3) with the restriction that

$$f:O(\gamma,R)\times O(\gamma,R)\to O(\gamma,R).$$

Then the equilibrium point γ of the equation (2.3) is a global attractor.

Proof: From Lemmas $2.i,\ i=12,13,14,15,\ we\ obtain\ x(t)\to\gamma\ when\ t\to\infty,$ then the equilibrium point γ of the equation (2.3) is a global attractor.

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