

Characterizations of Type-2 Harmonic Curvatures and General Helices in Euclidean space E^4

Faik Babadag

Department of MATHEMATICS, KIRIKKALE University, KIRIKKALE

Email: faik.babadag@kku.edu.tr

Abstract— In this study, we introduce quaternionic type-2 Harmonic Curvatures and General Helices according to Frenet frame in 4-dimensional Euclidean Space E^4 and investigate its properties for two cases. In the first case; we use a constant angle ϕ between a unit and fixed direction vector field U and the first relatively Frenet frame vector field V_1 of the curve, that is,

$$H(V_1, U) = \cos \phi = \text{const.}$$

where $h(V_1, U)$ is the real quaternion inner product. Since the relatively Frenet frame vector field V_1 of the curve makes a constant angle with the unit and fixed direction vector field U , we call this curve as a General helix in 4-dimensional Euclidean Space E^4 . And then, in the other case, we define new type-2 harmonic curvature functions and we give a vector field D which we call Darboux vector field for General helix. And then we obtain some characterizations for General helix in terms of type-2 harmonic curvature functions and the Darboux vector field D .

Keywords— Euclidean spaces, General helix, type-2 harmonic curvatures, Quaternion algebra, Quaternionic frame.

I. INTRODUCTION

The curves are a part of our lives are the indispensable. For example, heart chest film with X-ray curve, how to act is important to us. Curves give the movements of the particle in Physics.

Helical curves are very important type of curves. Because, helices are among the simplest objects in the art, molecular structures, nature, etc. For example, the path, aroused by the climbing of beans and the orbit where the progressing of the screw are a helix curves. Also, in medicine DNA molecule is formed as two intertwined helices and many proteins have helical structures, known as alpha helices. So, such curves are very important for understand to nature. Therefore, lots of author interested in the helices and they published many papers in Euclidean 3 and 4 - space [1-2],].

Helix curve is defined by the property that the tangent vector field makes a constant angle with a fixed direction. In 1802, M. A. Lancet first proposed a theorem and in 1845, B. de Saint Venant first proved this theorem: "A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant" [3-5].

In 1987, The Serret-Frenet formulae for quaternionic curves in $\mathbb{R}^3$ are introduced by K. Bharathi and M. Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the quaternionic curves in $\mathbb{R}^4$ by the formulae in $\mathbb{R}^3$, [6]. Then, lots of studies have been published by using this study. One of them is A. C. Çöken and A. Tuna's study [7] which they gave Serret-Frenet formulas, inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi- Euclidean spaces E_1^3 and E_2^4 .

In this work, we give quaternionic type-2 Harmonic Curvatures and General Helices according to Frenet frame in 4-dimensional Euclidean Space E^4 and investigate its properties. We use a constant angle ϕ between a unit and fixed direction vector field U and the first relatively Frenet frame vector field V_1 of the curve, that is,

$$H(V_1, U) = \cos \phi = \text{const.}$$

where $h(V_1, U)$ is the real quaternion inner product. Since the relatively Frenet frame vector field V_1 of the curve makes a constant angle with the unit and fixed direction vector field U , we call this curve as a General helix in 4-dimensional Euclidean Space E^4 . And then, we define new type - 2 harmonic curvature functions and we give a vector field D which we call Darboux vector field for General helix. And then we obtain some characterizations for General helix in terms of type-2 harmonic curvature functions and the Darboux vector field D .

II. PRELIMINARIES

Let Q_H be the four-dimensional vector space over a field H whose characteristic greater than 2. Let e_i ($1 \leq i \leq 4$) be a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i and extended to the whole of the vector space distributivity as follows [6]:

A real quaternion is defined by $q = ae_1 + be_2 + ce_3 + d$ (or $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$). Then a quaternion q can now write as $q = S_q + V_q$, where S_q and V_q are the scalar part and vectorial part of q , respectively. we define the set of all real quaternions by

$$Q_H = \{ q \mid q = ae^1 + be^2 + ce^3 + d ; a, b, c, d \in R \text{ and } e^1, e^2, e^3 \in R^3 \}.$$

Using these basic products we can now expand the product of two quaternions to give

$$p \times q = S_p S_q + \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q \text{ for every } p, q \in Q_H.$$

where we have used the dot and cross products in Euclidean space E^4 . We see that the quaternionic product contains all the products of Euclidean space E^4 . There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -ae_1 - be_2 - ce_3 \text{ for every } q = ae_1 + be_2 + ce_3 + d \in Q_H.$$

which is called the "Hamiltonian conjugation". This h-inner product of two quaternions is define by

$$h(p, q) = \frac{1}{2} [(p \times \gamma q) + (q \times \gamma p)] \text{ for every } p, q \in Q_H$$

where h is the symmetric, non-degenerate, real valued and bilinear form. The norm of real quaternion q is denoted by

$$\|q\|^2 = |h_{-}\{v\}(q, q)| = |(q \times \gamma q)| = |a^2b^2 + c^2 + d^2|$$

for $p, q \in Q_H$ where if $h(p, q) = 0$ then p and q are called h-orthogonal.

The concept of a spatial quaternion will be made use throughout our work. q is called a spatial quaternion whenever $q + \gamma q = 0$ [6, 7].

III. SERRET-FRENET FORMULAE FOR QUATERNIONIC CURVES EUCLIDEAN SPACE

Definition . The four-dimensional Euclidean spaces in E^4 are identified with the spaces of unit quaternions. Let

$$\alpha : I \subset \mathbb{R} \rightarrow Q_V$$

$$s \rightarrow \alpha(s) = \sum_{i=1}^4 \alpha_i(s) e_i, 1 \leq i \leq 4, e_4 = 1$$

be a smooth curve in E^4 . Let the parameter s be chosen such that the tangent $V_1(s) = \alpha'(s)$ has unit magnitude. Let $\{V_1; V_2; V_3; V_4\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean spaces E^4 . Then Frenet formulas are given by

$$\begin{cases} V_1'(s) = \kappa(s)V_2(s) \\ V_2'(s) = \tau(s)V_3(s) - \kappa(s)V_1(s) \\ V_3'(s) = -\tau(s)V_2(s) + \sigma(s)V_4(s) \\ V_4'(s) = -\sigma(s)V_3(s) \end{cases} \quad (1)$$

We may express Frenet formulae of the Frenet trihedron in the matrix form:

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \quad (2)$$

IV. TYPE-2 HARMONIC CURVATURES AND GENERAL HEICES IN EUCLIDEAN SPACE E^4

Definition . Let $\alpha(s)$ be a quaternionic curve in E^4 with arc-length parameter s . The type-2 harmonic curvatures of α are the functions, such that

$$G_i: I \rightarrow R, 1 \leq i \leq 4$$

$$G_i = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ \kappa/\tau & i = 3 \\ (1/\sigma)G_3' & i = 4 \end{cases}$$

and satisfying the condition

$$\frac{dG_4}{ds} = -\sigma G_3$$

Theorem. Let $\alpha = \alpha(s)$ be a quaternionic curves in Euclidean space E^4 with arc-length parameter s . Let $\{V_1; V_2; V_3; V_4\}$ and $\{G_1; G_2; G_3; G_4\}$ denote the Frenet frame and the type-2 harmonic curvatures of curve, respectively. Then α is a general helix if and only if the functions

$$G_1^2 + G_2^2 = c$$

is constant.

Proof. Let $\alpha = \alpha(s)$ be general helix in Euclidean space E^4 . Let U be the direction with which V_1 makes a constant angle ϕ and, without loss of generality, we suppose that $\langle U, U \rangle = 1$.

$$U = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 \quad (3)$$

that is,

$$\begin{cases} \lambda_1 = \langle V_1, U \rangle = \cos \phi = \text{constant} \\ \lambda_2 = \langle V_2, U \rangle \\ \lambda_3 = \langle V_3, U \rangle \\ \lambda_4 = \langle V_4, U \rangle \end{cases} \quad (4)$$

By taking the derivate of (3) with respect to s and using the Frenet formula, we have

$$\lambda_1'(s) = \kappa(s) \langle V_2(s), U(s) \rangle = \kappa(s) \lambda_2(s) = 0. \quad (5)$$

Then $\lambda_2 = 0$ and therefore,

$$U = \lambda_1 V_1 + \lambda_3 V_3 + \lambda_4 V_4$$

The differentiation of (5) gives,

$$(\kappa \lambda_1 - \tau \lambda_3) V_2 + (\lambda_3' - \sigma \lambda_4) V_3 + (\lambda_4' + \sigma \lambda_3) V_4 = 0$$

which implies

$$\begin{cases} \kappa \lambda_1 - \tau \lambda_3 = 0 \\ \lambda_3' - \sigma \lambda_4 = 0 \\ \lambda_4' + \sigma \lambda_3 = 0 \end{cases} \quad (6)$$

Define the function $G_i = G_i(s)$ as follows

$$\lambda_i(s) = G_i(s) \lambda_1, \quad 3 \leq i \leq 4. \quad (7)$$

We point out that $\lambda_1 \neq 0$: on the contrary, (6) gives $\lambda_i = 0$, for $3 \leq i \leq 4$ and so, $U = 0$ contradiction. From the equation (1), we get

$$\begin{cases} G_3 = (\kappa/\tau) \\ G_4 = (1/\sigma)G_3' \end{cases} \quad (8)$$

The last equation of (6) leads to the following condition:

$$G_4' + \sigma G_3 = 0. \quad (9)$$

In particular, and from last equation of (8)

$$G_3' + \sigma G_4 = 0. \quad (10)$$

If we take derivative of (9) and use the last equation, we get the second order differential equation

$$\begin{cases} G_4'' + \sigma'G_3 + \sigma G_3' = 0 \\ G_4'' + \sigma'(-1/\sigma)G_4 + \sigma(\sigma G_4) = 0 \\ G_4'' - ((\sigma')/\sigma)G_4' + \sigma^2 G_4 = 0. \end{cases} \quad (11)$$

We do change of variables:

$$t(s) = \int^s \sigma(s)ds, \quad \frac{dt}{ds} = \sigma(s).$$

Then the equation (11) becomes

$$G_4''(t) + G_4(t) = 0.$$

The general solution of this equation is obtained as

$$G_4(t(s)) = A \cos \int^s \sigma(s)ds + B \sin \int^s \sigma(s)ds \quad (12)$$

where A and B are constants. From equation (9), the function G_3 is given by

$$\begin{cases} G_4(t(s)) = -(1/\sigma)G_4' \\ = -(1/\sigma)(-(1/\sigma)A \cos \int^s \sigma(s)ds + (1/\sigma)B \sin \int^s \sigma(s)ds) \\ A \sin \int^s \sigma(s)ds - B \cos \int^s \sigma(s)ds. \end{cases} \quad (13).$$

From equations (12) and (13), we have

$$G_3^2 + G_4^2 = A^2 + B^2 = c.$$

Conversely, assume that the condition (2) is satisfied for a curve α . Define the unit vector U by

$$U = (V_1 + G_3V_3 + G_4V_4)\cos\phi.$$

By taking (8), (9) and (10), a differentiation of U given that

$$\frac{dU}{ds} = [(\kappa - G_3\tau)V_2 + (G_3' - G_4\sigma)V_3 + (G_4' + G_3\sigma)V_4]\cos\phi = 0$$

which it means that U is a constant vectors. On the other hand, the inner product between the unit tangent vector V_1 with U is

$$\langle V_1, U \rangle = \cos\phi.$$

Thus α is a general helix curve and the proof is completed.

Lemma. Let $\alpha(s)$ be a unit speed general helix in Euclidean space E^4 . Let $\{V_1; V_2; V_3; V_4\}$ and $\{G_1; G_2; G_3; G_4\}$ denote the Frenet frame and the type-2 harmonic curvatures of curve, respectively. Then, the following equations holds:

$$\begin{cases} \langle V_1, U \rangle = G_1 \langle V_1, U \rangle \\ \langle V_2, U \rangle = G_2 \langle V_2, U \rangle \\ \langle V_3, U \rangle = G_3 \langle V_3, U \rangle \\ \langle V_4, U \rangle = G_4 \langle V_4, U \rangle \end{cases} \quad (14)$$

where U is an axis of the general helix α . By using the above Lemma, we have the following corollary:

Corollary. If U is an axis of the general helix α , then we can write

$$U = (G_1V_1 + G_2V_2 + G_3V_3 + G_4V_4)\cos\phi.$$

Definition. Let $\alpha(s)$ be a non-degenerate unit speed curve in Euclidean space E^4 . Let $\{V_1; V_2; V_3; V_4\}$ and $\{G_1; G_2; G_3; G_4\}$ denote the Frenet frame and the type-2 harmonic curvatures of curve, respectively, the vector

$$D = G_1V_1 + G_2V_2 + G_3V_3 + G_4V_4 \quad (15)$$

is called the type-2 Darboux vector of the curve α . Also,

$$D = G_1V_1 + G_3V_3 + G_4V_4$$

is an axis of the general helix α .

Lemma. Let $\alpha(s)$ be a non-degenerate unit speed curve in Euclidean space E^4 . Let $\{V_1; V_2; V_3; V_4\}$ and $\{G_1; G_2; G_3; G_4\}$ denote the Frenet frame and the type-2 harmonic curvatures of curve, respectively, then α is a general helix if and only if D is constant vector.

Proof. Let $\alpha(s)$ be a general helix in Euclidean space E^4 . From corollary 1,

$$D = G_1V_1 + G_3V_3 + G_4V_4.$$

By differentiating the D with respect to s

$$D' = (\kappa - G_3\tau)V_2 + (G_3' - G_4\sigma)V_3 + (G_4' + G_3\sigma)V_4.$$

By using equation (8),(9) and (10), we get $D' = 0$. Therefore, D is constant vector.

Converaly, if D is constant vector. Then we can see that

$$\langle D, V_1 \rangle = \langle V_1 + G_3V_3 + G_4V_4, V_1 \rangle$$

$$\langle D, V_1 \rangle = \langle V_1, V_1 \rangle = 1.$$

Thus we get

$$\cos\phi = \frac{\langle D, V_1 \rangle}{\|D\| \|V_1\|} = \frac{1}{\|D\|}$$

where ϕ is constant angle between D and V_1 . In this case, we can define a unique axis of the general helix such that :
 $U = D\cos\phi$. From this equation

$$\begin{aligned} \langle U, V_1 \rangle &= \langle D, V_1 \rangle \cos\phi \\ &= \cos\phi = \text{constant.} \end{aligned}$$

Therefore, U is a constant. So, this complete the proof.

Lemma. In three- dimensional Euclidean space, from equation (15), we can write the axis of a non-degenerate curve as:

$$\begin{aligned} D &= G_1 t + G_2 n + G_3 b \\ &= G_1 t + G_3 b \\ &= t + (\kappa/\tau) b \end{aligned}$$

where $\{t; n; b\}$ and $\{\kappa; \tau\}$ are the Frenet apparatus and with non-zero curvatures in the Euclidean spaces E^3 , respectively. If we take derivative of D along the curve, we get

$$\begin{aligned} D' &= t' + ((\kappa/\tau))' b + (\kappa/\tau) b' \\ &= \kappa n + ((\kappa/\tau))' b + (\kappa/\tau)(-\tau n) \\ &= ((\kappa/\tau))' b. \end{aligned}$$

Thus, from the equation, if the curve is a general helix, then from Lemma 2, we have

$$D' = 0$$

$((\kappa/\tau))' = 0$, (κ/τ) is constant, therefore the curve is a general helix.

Lemma. There are no general helices with non-zero constant curvatures (i.e., W -curve) in Euclidean space E^4 .

Proof. In four-dimensional Euclidean space, from equation (15), we get

$$\begin{aligned} D &= G_1 V_1 + G_2 V_2 + G_3 V_3 + G_4 V_4 \\ D &= V_1 + (\kappa/\tau) V_3 + (1/\sigma)(\kappa/\tau) V_4 \end{aligned}$$

where κ, τ and σ are curvatures of the curve. If all the curvatures of the curve are non-zero constants, then the curve is a W -curve, then we get

$$D = V_1 + (\kappa/\tau) V_3. \quad (16)$$

If we take derivative of equation (16), we obtain

$$\begin{aligned} D' &= \kappa V_1 + (\kappa/\tau)(-\tau V_2 + \sigma V_4) \\ &= (\kappa/\tau) \sigma V_4. \end{aligned}$$

So, we can easily see that D' is not equal to zero, then D is not constant vector. In this case, according to Lemma 2 the curve is not general helix.

Lemma. There are no general helices with non-zero constant curvature ratios (i.e., CCr -Curve) in E^4 .

Proof. The proof of this Lemma is the same as the proof of Lemma 1.

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