Analytical Solution with Two Time Scales of Circular Restricted Three-Body Problem

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Abstract— *Analytical solution performs a vital role in a wide variety of deep space exploration missions, especially the periodic solutions which were believed to be the unique avenue to solve three-body problem by Poincaré. As the absence of a general solution for the problem, an approximate analytical solution of the circular restricted three-body problem is addressed by employing multiple scales method in conjunction with some analytical techniques. It is worthwhile to note that the presented solution in three-dimensional space contains two time scales: t and εt (ε is a small, dimensionless parameter), which is significative to improve and perfect the known literature.*

Keywords— Analytical solution, multiple scales, three-body problem.

I. INTRODUCTION

An existence of conditional periodic solutions of circular restricted three-body problem (CR3BP) was presented by Gao [1]. However, because of lacking exact analytical solutions for the problem, a considerable amount of researchers were attracted to develop the approximate analytical solutions of the problem. In 1973, Farquhar and Kamel [2] developed an approximation method for this type of orbit. Richardson [3, 4] presented a third-order analytical solution about the collinear libration points of the CR3BP based on the method of successive approximations and a technique similar to the Lindstedt-Poincaré method. Lu and Zhao [5] put forward a kind of improved third-order approximate analytical solution in 2009, which is more accurate than the classical analytic solution of Richardson. In addition, Nayfeh [6, 7] studied two types of resonance near the planar triangular libration points under a case of small amplitude. The planar 3:1 resonance with time scales t and $\varepsilon^2 t$ was discussed when the planar restricted three-body problem was expanded to second-order terms in ε. Moreover, approximate analytical solutions of the planar 2:1 resonance with time scales t and εt was also investigated when the problem was expanded to third-order terms in ε. These creative results provide us with new motivation in studying the solutions of the three-body problem.

In this paper, multiple scales method (see Nayfeh [8]) will be employed to construct a three-dimensional (3D) approximate analytical solution of a spatial CR3BP. The addressed 3D solution will be written in the following form

$$\mathbf{u} = \varepsilon \mathbf{u}_1(t, \varepsilon t) + \varepsilon^2 \mathbf{u}_2(t, \varepsilon t), \tag{1}$$

where $\mathbf{u} = (x, y, z)$, $\mathbf{u}_i = (x_i, y_i, z_i), i = 1, 2$, and ε is a small, dimensionless parameter.

It is worthwhile to note that when the problem is expanded to third-order, the above presented solution in 3D space contains two time scales, which is significative to improve and perfect the known literature, in which the constructed solutions mainly possess three cases: a) The three-body problem discussed is a planar case, so the proposed approximate solution is a two-dimensional curve. b) The 3D approximate solution contains only one time scale when the spatial CR3BP was expanded to the third-order. c) The 3D approximate solution contains two time scales when the spatial CR3BP was expanded to the second-order.

II. CONSTRUCTION OF THE 3D APPROXIMATE ANALYTICAL SOLUTION

Note that the third-order approximate system with small amplitude of the CR3BP in rotating frame can be described as follows (the general third-order approximate system can be found at Koon et al. [9])

$$\ddot{x} - 2\dot{y} - (1 + 2c_2)x = \frac{3}{2}\varepsilon c_3\left(2x^2 - y^2 - z^2\right) + 2\varepsilon c_4\left(2x^2 - 3y^2 - 3z^2\right), \tag{2a}$$

$$\ddot{y}+2\dot{x}+(c_2-1)y=-3\varepsilon c_3 xy-\frac{3}{2}\varepsilon c_4 y\left(4x^2-y^2-z^2\right), \tag{2b}$$

$$\ddot{z}+c_2 z=-3\varepsilon c_3 xz-\frac{3}{2}\varepsilon c_4 z\left(4x^2-y^2-z^2\right), \tag{2c}$$

where

1. $c_n=\frac{1}{\gamma_L^3}\left[(\pm 1)^n\mu+(-1)^n\frac{(1-\mu)\gamma_L^{n+1}}{(1\mp\gamma_L)^{n+1}}\right]$, the upper sign is for L_1 and the lower one for L_2,

$c_n=\frac{1}{\gamma_L^3}\left[1-\mu+\frac{\mu\gamma_L^{n+1}}{(1+\gamma_L)^{n+1}}\right]$, for libration point L_3,

$\gamma_L=n_1^{\frac{2}{3}}$, n_1 is the angular velocity of relative movement between two primaries.

The solution of equations (2a) ~ (2c) are assumed to possess the following form

$$x=\varepsilon x_1(T_0,T_1)+\varepsilon^2 x_2(T_0,T_1), \tag{3a}$$

$$y=\varepsilon y_1(T_0,T_1)+\varepsilon^2 y_2(T_0,T_1), \tag{3b}$$

$$z=\varepsilon z_1(T_0,T_1)+\varepsilon^2 z_2(T_0,T_1), \tag{3c}$$

where $T_0=t, T_1=\varepsilon t$.

Then, time derivatives become $\frac{d}{dt}=D_0+\varepsilon D_1+\cdots$, $\frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\cdots$, $D_k=\frac{\partial}{\partial T_k}$, $k=0,1,\cdots$.

Substituting equations (3) into equations (2a) ~ (2c) and equating the coefficients of ε, ε^2 , and ε^3 to zero, leads to

$$D_0^2 x_1-2D_0 y_1-(1+2c_2)x_1=0, \tag{4a}$$

$$D_0^2 y_1+2D_0 x_1+(c_2-1)y_1=0, \tag{4b}$$

$$D_0^2 z_1+c_2 z_1=0. \tag{4c}$$

$$D_0^2 x_2+2D_0 D_1 x_1-2D_0 y_2-2D_1 y_1-(1+2c_2)x_2=0, \tag{5a}$$

$$D_0^2 y_2+2D_0 D_1 y_1+2D_0 x_2+2D_1 x_1+(c_2-1)y_2=0, \tag{5b}$$

$$D_0^2 z_2+2D_0 D_1 z_1+c_2 z_2=0. \tag{5c}$$

$$2D_0 D_1 x_2-2D_1 y_2=\frac{3}{2}c_3\left(2x_1^2-y_1^2-z_1^2\right), \tag{6a}$$

$$2D_0D_1y_2 + 2D_1x_2 = -3c_3x_1y_1, \tag{6b}$$

$$2D_0D_1z_2 = -3c_3x_1z_1. \tag{6c}$$

Let secular terms be equal to zero, the general solution of equations (4a) ~ (4c) and (5a) ~ (5c) read

$$x_1 = \rho_1 \cos(\lambda t), \tag{7a}$$

$$y_1 = -\delta\rho_1 \sin(\lambda t), \tag{7b}$$

$$z_1 = \rho_2 \cos(\lambda t). \tag{7c}$$

$$x_2 = \alpha_1(T_1)\cos(\lambda t + \beta_1(T_1)), \tag{8a}$$

$$y_2 = -\delta\alpha_1(T_1)\sin(\lambda t + \beta_1(T_1)), \tag{8b}$$

$$z_2 = \alpha_2(T_1)\cos(\lambda t + \beta_2(T_1)), \tag{8c}$$

where ρ_1, ρ_2 and δ are constants, λ is the mould of the pure imaginary roots of equations (4), α_i and β_i ($i = 1, 2$) satisfy

$$\alpha_1'(T_1) = \frac{3}{8(\delta-\lambda)}c_3\left[\left(\left(2+\delta^2\right)\rho_1^2 - \rho_2^2\right)\cos(2\lambda t) + \left(2-\delta^2\right)\rho_1^2 - \rho_2^2\right]\sin(\lambda t + \beta_1(T_1))$$

$$+\frac{3}{4(1-\lambda\delta)}c_3\delta\rho_1^2\sin(2\lambda t)\cos(\lambda t + \beta_1(T_1)), \tag{9a}$$

$$\alpha_1\beta_1'(T_1) = \frac{3}{8(\delta-\lambda)}c_3\left[\left(\left(2+\delta^2\right)\rho_1^2 - \rho_2^2\right)\cos(2\lambda t) + \left(2-\delta^2\right)\rho_1^2 - \rho_2^2\right]\cos(\lambda t + \beta_1(T_1))$$

$$-\frac{3}{4(1-\lambda\delta)}c_3\delta\rho_1^2\sin(2\lambda t)\sin(\lambda t + \beta_1(T_1)), \tag{9b}$$

$$-2\lambda\left[\alpha_2'\sin(\lambda T_0 + \beta_2) + \alpha_2\beta_2'\cos(\lambda T_0 + \beta_2)\right] = -\frac{3}{2}c_3\delta\rho_1\rho_2[\cos(2\lambda T_0) + 1]. \tag{9c}$$

Therefore, a third-order approximate analytical solution with two time scales to equations (2a) ~ (2c) can be represented as

$$x = \varepsilon\rho_1\cos(\lambda t) + \varepsilon^2\alpha_1(\varepsilon t)\cos(\lambda t + \beta_1(\varepsilon t)), \tag{10a}$$

$$y = -\varepsilon\delta\rho_1\sin(\lambda t) - \varepsilon^2\delta\alpha_1(\varepsilon t)\sin(\lambda t + \beta_1(\varepsilon t)), \tag{10b}$$

$$z = \varepsilon\rho_2\cos(\lambda t) + \varepsilon^2\alpha_2(\varepsilon t)\cos(\lambda t + \beta_2(\varepsilon t)), \tag{10c}$$

where α_1 and β_1 satisfy equations (9a) and (9b), α_2 and β_2 satisfy equation (9c).

III. CONCLUSION

Since the governing equations of the third body is time-dependent in inertial frame, which appear as a high-dimensional nonlinear autonomous system. According to the abundant known literature, this system seems unlikely to be solved via analytical approaches. For example, if we employ multiple scales method, then it is difficult solving the solutions to the equations derived from equating coefficients of ε^2 to zero, not to mention the non-autonomous obtained from third-order terms in ε . However, these will be possible in rotating frame, where the equations are characterized by nonlinear autonomous system. In this frame, the method of multiple scales in conjunction with some analytical techniques is adopted to construct a 3D approximate analytical periodic solution for CR3BP. The solution was demonstrated with two different time scales t and εt when the system was expanded to ε^3 -order. This result improves and perfects the known literature.

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