

Reliability Evaluation of Multicommodity Limited-Flow Networks with Budget Constraints

Jsen-Shung Lin

Department of Information Management, Central Police University, Taiwan

E-mail: jslin168@mail.cpu.edu.tw

Abstract—Many real-world systems such as manufacturing systems, transportation systems and logistics/distribution systems that play important roles in our modern society can be regarded as multicommodity flow networks whose arcs have independent, finite and multi-valued random capacities. Such a flow network is a multistate system with multistate components and its reliability for level $(\mathbf{d};c)$, i.e., the probability that k different types of commodity can be transmitted from the source node to the sink node such that the demand level $\mathbf{d} = (d_1, d_2, \dots, d_k)$ is satisfied and the total transmission cost is less than or equal to c , can be evaluated in terms of minimal path vectors to level $(\mathbf{d};c)$ (named $(\mathbf{d};c)$ -MPs here). The main objective of this paper is to present an intuitive algorithm to generate all $(\mathbf{d};c)$ -MPs of such a flow network for each level $(\mathbf{d};c)$ in terms of minimal pathsets. Two examples are given to illustrate how all $(\mathbf{d};c)$ -MPs are generated by our algorithm and then the reliability of one example is computed.

Keywords— Reliability, limited-flow network, multicommodity, multistate system, $(\mathbf{d};c)$ -MP.

I. INTRODUCTION

Reliability is an important performance indicator in the planning, designing, and operation of a real-world system. Traditionally, it is assumed that the system under study is represented by a probabilistic graph in a binary-state model, and the system operates successfully if there exists one or more paths from the source node s to the sink node t . In such a case, reliability is considered as a matter of connectivity only and so it does not seem to be reasonable as a model for some real-world systems. Many physical systems such as manufacturing systems, transportation systems, and logistics/distribution systems can be regarded as flow networks in which arcs have independent, finite, and integer-valued random capacities. To evaluate the system reliability of such a flow network, different approaches have been presented [7, 9, 14-23, 26-28]. However, these models have assumed that the flow along any arc consisted of a single commodity only. For such a flow network with multicommodity, it is very practical and desirable to compute its reliability for level $(\mathbf{d};c)$, i.e., the probability that k different types of commodity can be transmitted from the source node to the sink node in the way that the demand level $\mathbf{d} = (d_1, d_2, \dots, d_k)$ is satisfied and the total transmission cost is less than or equal to c .

In general, reliability evaluation can be carried out in terms of minimal pathsets (MPs) in the binary state model case and $(\mathbf{d};c)$ -MPs (i.e., minimal path vectors to level $(\mathbf{d};c)$ [3], lower boundary points of level $(\mathbf{d};c)$ [12], or upper critical connection vector to level $(\mathbf{d};c)$ [7]) for each level $(\mathbf{d};c)$ in the multistate model case. The multicommodity limited-flow network with budget constraints here can be treated as a multistate system of multistate components and so the need of an efficient algorithm to search for all of its $(\mathbf{d};c)$ -MPs arises. The main purpose of this article is to present a simple algorithm to generate all $(\mathbf{d};c)$ -MPs of such a network in terms of minimal pathsets. Two examples are given to illustrate how all $(\mathbf{d};c)$ -MPs are generated and the reliability of one example is calculated by further applying the state-space decomposition method [4].

II. BASIC ASSUMPTIONS

Let $G = (N, A, U)$ be a directed limited-flow network with the unique source s and the unique sink t , where N is the set of nodes, $A = \{a_i \mid 1 \leq i \leq n\}$ is the set of arcs, and $U = (u_1, u_2, \dots, u_n)$, where u_i denotes the maximum capacity of each arc a_i for $i = 1, 2, \dots, n$. Such a flow network is assumed to further satisfy the following assumptions:

1. Each node is perfectly reliable. Otherwise, the network will be enlarged by treating each of such nodes as an arc [1].
2. The capacity of each arc a_i is an integer-valued random variable that takes integer values from 0 to u_i according to a given distribution.

3. Every unit flow of commodity ℓ consumes a given amount ρ^ℓ of the capacity associated with each arc.
4. The capacities of different arcs are statistically independent.
5. Flow in the network must be integer-valued and satisfy the so-called flow-conservation law [10]. This means that no flow will disappear or be created during the transmission.

Assumption 4 is made just for convenience. If it fails in practice, the proposed algorithm to search for all $(\mathbf{d};c)$ -MPs is still valid except that the reliability computation in terms of such $(\mathbf{d};c)$ -MPs should take the joint probability distributions of all arc capacities into account.

Since there are k different types of commodity within the network, the system demand level can be represented as a k -tuple vector $\mathbf{d} = (d_1, d_2, \dots, d_k)$ where d_j is the demand level of commodity j for $j = 1, 2, \dots, k$. Let $X = (x_1, x_2, \dots, x_n)$ be a system-state vector (i.e., the current capacity of each arc a_i under X is x_i , where x_i takes integer values $0, 1, 2, \dots, u_i$), and $V(X) = (V(X)_1, V(X)_2, \dots, V(X)_k)$, the system maximal flow vector under X where $V(X)_j$ denotes the maximal flow of commodity j under X . (Whenever $k \geq 2$, there may be more than one maximal flow vector for each X . See the Appendix for more details.) Under the system-state vector $X = (x_1, x_2, \dots, x_n)$, the arc set A has the following three important subsets:

$N_X = \{a_i \in A \mid x_i > 0\}$, $Z_X = \{a_i \in A \mid x_i = 0\}$, and $S_X = \{a_i \in N_X \mid V(X - e_i) < V(X)\}$, where $e_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$, with $\delta_{ij} = 1$ if $j = i$ and 0 if $j \neq i$. In fact, $A = S_X \cup (N_X \setminus S_X) \cup Z_X$ is a disjoint union of A under X .

A system-state vector X is said to be a $(\mathbf{d};c)$ -MP if and only if: (1) its system capacity level is $\mathbf{d}$ (i.e., $V(X) = \mathbf{d}$), (2) each nonzero-capacity arc under X is sensitive (i.e., $N_X = S_X$), and (3) the total transmission cost is less than or equal to c . If level $(\mathbf{d};c)$ is given, then the probability that k different types of commodity can be transmitted from the source node to the sink node in the way that the demand level $\mathbf{d} = (d_1, d_2, \dots, d_k)$ is satisfied and the total transmission cost is less than or equal to c , is taken as the system reliability.

III. MODEL BUILDING

Suppose that $P^1, P^2, \dots, P^m$ are the collection of all MPs of the system, and let $C = (c_1^1, c_1^2, \dots, c_1^k, c_2^1, c_2^2, \dots, c_2^k, \dots, c_n^1, c_n^2, \dots, c_n^k)$ denote the transmission cost vector where c_i^ℓ is the unit transmission cost of commodity ℓ through arc a_i . For each P^j , $W_j^\ell = \sum_i \{c_i^\ell \mid a_i \in P^j\}$ and $L_j = \min\{u_i \mid a_i \in P^j\}$ are taken as the unit transmission cost of commodity ℓ and maximum capacity through it, respectively. Under the flow-conservation law, any feasible flow pattern from s to t should satisfy that (1) the total flow-in and the total flow-out of each commodity for any given node (except for s and t) are equal, and (2) every unit flow of each commodity from s to t should travel through one of the MPs. Hence, under the system-state vector $X = (x_1, x_2, \dots, x_n)$ with $V(X) = (d_1, d_2, \dots, d_k)$, any feasible flow pattern that the total transmission cost is less than or equal to c can be represented as a flow vector $(f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ where f_j^ℓ is the flow of commodity ℓ transmitted through P^j such that the following four conditions are satisfied:

$$\sum_{j=1}^m f_j^\ell = d_\ell \text{ for each } \ell = 1, 2, \dots, k \quad (1)$$

$$\sum_{\ell=1}^k f_j^\ell \rho^\ell \leq L_j \text{ for each } j = 1, 2, \dots, m \quad (2)$$

$$\sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \leq u_i \text{ for each } i = 1, 2, \dots, n \quad (3)$$

$$\sum_{\ell=1}^k \sum_{j=1}^m W_j^\ell f_j^\ell \leq c \quad (4)$$

Note that $\sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\}$ is the least amount of capacity needed for a_i under such a flow pattern

$(f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$, and so, under the system-state vector X , $\sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\}$

does not exceed the current capacity x_i of a_i . This fact is given in the following theorem.

Theorem 1. Let $X = (x_1, x_2, \dots, x_n)$ be any system-state vector for which $V(X) = \mathbf{d}$. Then, the following is a necessary condition for the flow-conservation law to hold under X :

$$x_i \geq \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \text{ for each } i = 1, 2, \dots, n \quad (5)$$

for any $(f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ which is a feasible flow pattern of flow $\mathbf{d}$ under X .

Theorem 2. Let X be a $(\mathbf{d};c)$ -MP. Then, the following is a necessary condition for the flow-conservation law to hold under X :

$$x_i = \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \text{ for each } i = 1, 2, \dots, n \quad (6)$$

for any $(f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ which is a feasible flow pattern of flow $\mathbf{d}$ under X .

Proof . By Theorem 1, $x_i \geq \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\}$ for each $i = 1, 2, \dots, n$.

1. For each $a_i \in Z_X$, $x_i = 0$ and so (6) holds.

2. It remains to show that (6) holds for each $a_i \in N_X$. Suppose, on the contrary, that there exists an arc $a_i \in N_X$ such that

$$\sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} < x_i. \text{ Then, } \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \leq x_i - 1. \text{ In particular, } V(X - e_i) = \mathbf{d} = V(X), \text{ and so}$$

$$a_i \notin S_X, \text{ which contradicts to the fact that } X \text{ is a } (\mathbf{d};c)\text{-MP. Hence, } x_i = \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \text{ for each } a_i \in N_X.$$

The vector $X = (x_1, x_2, \dots, x_n)$ obtained by first solving $F = (f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ subject to constraints (1) - (4) and then transforming such $F = (f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ to $X = (x_1, x_2, \dots, x_n)$ by applying the relationship in (6), will be taken as a $(\mathbf{d};c)$ -MP candidate. To make it clearer that all $(\mathbf{d};c)$ -MPs can be generated by the proposed method, the following theorem is necessary.

Theorem 3. Every $(\mathbf{d};c)$ -MP is a $(\mathbf{d};c)$ -MP candidate.

Proof. Let $X = (x_1, x_2, \dots, x_n)$ be any $(\mathbf{d};c)$ -MP. By definition, we know that the maximal flow from s to t under X is $\mathbf{d}$ (i.e., $V(X) = \mathbf{d}$) and the total transmission cost is less than or equal to c . Hence, under the system-state vector X , there exists at least one feasible flow pattern $F = (f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ of flow $\mathbf{d} = (d_1, d_2, \dots, d_k)$ such that conditions (1) - (4) are satisfied. As $X = (x_1, x_2, \dots, x_n)$ is a $(\mathbf{d};c)$ -MP, we thus conclude, by Theorem 2, that

$$x_i = \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \text{ for each } i = 1, 2, \dots, n. \text{ This means that } X \text{ is a } (\mathbf{d};c)\text{-MP candidate. Hence, every } (\mathbf{d};c)\text{-MP is a}$$

$(\mathbf{d};c)$ -MP candidate.

In this article, we first find feasible solutions $F = (f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ subject to constraints (1) - (4) by applying an implicit enumeration method (e.g., backtracking or branch-and-bound [11]) and then transform such integer-valued solutions into (d;c)-MP candidates $(x_1, x_2, \dots, x_n)$ via the relationship in (6). Each (d;c)-MP candidate X must be checked whether all nonzero-capacity arcs under X (i.e., $arc \in N_X$) belong to S_X . If the answer is “yes”, then X is a (d;c)-MP. Otherwise, X is not a (d;c)-MP. The following two theorems play the crucial roles in checking whether a (d;c)-MP candidate is a (d;c)-MP.

Theorem 4. For each (d;c)-MP candidate X , there exists at least one (d;c)-MP Y such that $Y \leq X$. In particular, X is not a (d;c)-MP if such a Y satisfies $Y < X$ (where $Y \leq X$ if and only if $y_i \leq x_i$ for $i=1, 2, \dots, n$ and $Y < X$ if and only if $Y \leq X$ and $y_i < x_i$ for at least one i).

Proof. If X is a (d;c)-MP, then Y must be taken as X . Suppose that X is not a (d;c)-MP; then there exists a nonzero-capacity arc a_i (i.e., $a_i \in N_X$) such that $V(X - e_i) = V(X) = \mathbf{d}$. Let $X^1 = X - e_i$. Suppose that X^1 is a (d;c)-MP; then Y is taken as X^1 . Otherwise, the same procedure may be repeated for X^1 . However, this procedure will stop in finite steps, i.e., there exists an integer p such that $X^p \leq X^{p-1} \leq \dots \leq X^1 \leq X$ with $V(X^p) = \mathbf{d}$ and $N_{X^p} = S_{X^p}$. The proof is thus concluded by letting $Y = X^p$.

Theorem 5. If the network is acyclic (i.e., contains no directed cycle), then each (d;c)-MP candidate is a (d;c)-MP.

Proof. Let $X = (x_1, x_2, \dots, x_n)$ be any (d;c)-MP candidate. By Theorem 4, we know that there exists a (d;c)-MP $Y = (y_1, y_2, \dots, y_n)$ such that $Y \leq X$. Since $V(X - Y) = \mathbf{d} - \mathbf{d} = 0$, no flow is transmitted from s to t under $X - Y = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$. Hence, in case $X \neq Y$, $I = \{i \mid x_i - y_i > 0\}$ is not empty and so $\{a_i \mid i \in I\}$, which is a subset of N_X , must form cycles since the flow conserves at each node (except for s and t) and there is no other sink except t (see Ford and Fulkerson [10] or Ahuja et al. [2] for more details). This means that if the network is acyclic, then $I = \emptyset$ and so $X = Y$, i.e., each (d;c)-MP candidate X is a (d;c)-MP.

Suppose that $X^1, X^2, \dots, X^q$ are total (d;c)-MP candidates. We can thus conclude, by Theorem 4, that X^j is a (d;c)-MP if $X^j \not\leq X^i$ for all $j=1, 2, \dots, q$ but $j \neq i$.

IV. ALGORITHM

Suppose that all MPs, $P^1, P^2, \dots, P^m$, have been stipulated in advance [5-6, 24-25], the family of all (d;c)-MPs can then be derived by the following steps:

Step 1. For each $P^j (j=1, 2, \dots, m)$, calculate $L_j = \min\{u_i \mid a_i \in P^j\}$ and $W_j^\ell = \sum_i \{c_i^\ell \mid a_i \in P^j\}$

Step 2. Find all feasible solutions $F = (f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ subject to the following constraints by applying an implicit enumeration method:

- (1) $\sum_{j=1}^m f_j^\ell = d_\ell$ for each $\ell=1, 2, \dots, k$
- (2) $\sum_{\ell=1}^k f_j^\ell \rho^\ell \leq L_j$ for each $j=1, 2, \dots, m$
- (3) $\sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \leq u_i$ for each $i=1, 2, \dots, n$
- (4) $\sum_{\ell=1}^k \sum_{j=1}^m W_j^\ell f_j^\ell \leq c$

where f_j^ℓ is a nonnegative integer for $j = 1, 2, \dots, m$ and $\ell = 1, 2, \dots, k$.

Step 3. Transform the solutions $(f_1^1, f_1^2, \dots, f_1^k, f_2^1, f_2^2, \dots, f_2^k, \dots, f_m^1, f_m^2, \dots, f_m^k)$ into (d;c)-MP candidates

$$X = (x_1, x_2, \dots, x_n) \text{ via } x_i = \sum_{\ell=1}^k \sum_{j=1}^m \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \text{ for } i = 1, 2, \dots, n.$$

Step 4. Check each candidate X one at a time whether it is a (d;c)-MP:

(A) If the network is acyclic, then each candidate is a (d;c)-MP.

(B) If the network is cyclic, and suppose $\{X^1, X^2, \dots, X^q\}$ is the family of all such (d;c)-MP candidates, then X^i is a (d;c)-MP if $X^j \not\prec X^i$ for all $j = 1, 2, \dots, q$ but $j \neq i$.

V. EXAMPLES

The following two examples are used to illustrate the proposed algorithm:

Example 1.

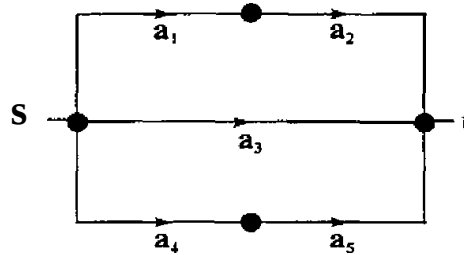


FIG. 1. A SERIES-PARALLEL NETWORK.

Consider the network in Fig. 1. It is known that $U = (u_1, u_2, u_3, u_4, u_5) = (1, 2, 2, 2, 1)$, $C = (2, 3, 2, 3, 5, 6, 2, 3, 2, 3)$, $\rho = (\rho_1, \rho_2) = (1, 2)$, and there exists three MPs; $P^1 = \{a_1, a_2\}$, $P^2 = \{a_3\}$, $P^3 = \{a_4, a_5\}$. Given $\mathbf{d} = (1, 1)$ and $c = 12$, the family of (d;c)-MPs is derived as follows:

Step 1. $L_1 = \min\{1, 2\} = 1$, $L_2 = \min\{2\} = 2$, $L_3 = \min\{2, 1\} = 1$, $W_1^1 = 2 + 2 = 4$, $W_1^2 = 3 + 3 = 6$, $W_2^1 = 5$, $W_2^2 = 6$, $W_3^1 = 2 + 2 = 4$, $W_3^2 = 3 + 3 = 6$.

Step 2. Find all feasible solutions $(f_1^1, f_1^2, f_2^1, f_2^2, f_3^1, f_3^2)$ subject to the following constraints by applying an implicit enumeration method:

$$\begin{cases} f_1^1 + f_2^1 + f_3^1 = 1 \\ f_1^2 + f_2^2 + f_3^2 = 1 \\ \begin{cases} f_1^1 \times 1 + f_1^2 \times 2 \leq 1 \\ f_2^1 \times 1 + f_2^2 \times 2 \leq 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \leq 1 \end{cases} \\ \begin{cases} f_1^1 \times 1 + f_1^2 \times 2 \leq 1 \\ f_1^1 \times 1 + f_1^2 \times 2 \leq 2 \\ f_2^1 \times 1 + f_2^2 \times 2 \leq 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \leq 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \leq 1 \end{cases} \end{cases}$$

$$4f_1^1 + 6f_1^2 + 5f_2^1 + 6f_2^2 + 4f_3^1 + 6f_3^2 \leq 12$$

where f_j^ℓ is a nonnegative integer for $j = 1, 2, 3$ and $\ell = 1, 2$.

Total feasible solutions are $F^1 = (1, 0, 0, 1, 0, 0)$ and $F^2 = (0, 0, 0, 1, 1, 0)$.

Step 3. Transform such feasible solutions into $(\mathbf{d};\mathbf{c})$ -MP candidates $X = (x_1, x_2, x_3, x_4, x_5)$ via

$x_i = \sum_{\ell=1}^3 \sum_j \{f_j^\ell \rho^\ell \mid a_i \in P^j\}$ for $i = 1, 2, \dots, 5$. Then $X^1 = (1, 1, 2, 0, 0)$ and $X^2 = (0, 0, 2, 1, 1)$ are total $(\mathbf{d};\mathbf{c})$ -MP candidates.

Step 4. The network is acyclic, and $\{X^1, X^2\}$ is the family of all $(\mathbf{d};\mathbf{c})$ -MP candidates. Since $X^i \not\prec X^j$, $X^1 = (1, 1, 2, 0, 0)$ and $X^2 = (0, 0, 2, 1, 1)$ are total $(\mathbf{d};\mathbf{c})$ -MP

Example 2.

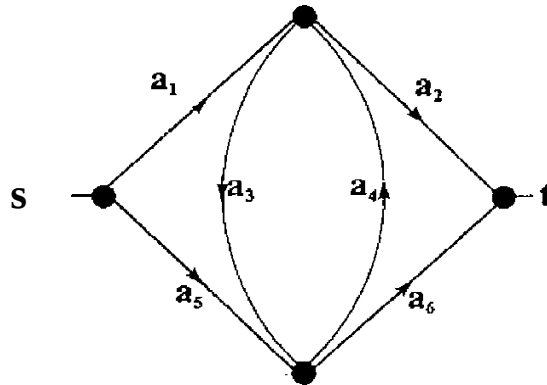


FIG. 2. A BRIDGE NETWORK.

TABLE 1
PROBABILITY DISTRIBUTIONS OF ARC CAPACITIES IN EXAMPLE 2

Arc	Capacity	Probability	Arc	Capacity	Probability
a_1	3	0.60	a_4	1	0.90
	2	0.25		0	0.10
	1	0.10	a_5	2	0.80
	0	0.05		1	0.15
a_2	2	0.70		0	0.05
	1	0.20	a_6	3	0.65
	0	0.10		2	0.20
a_3	1	0.90		1	0.10
	0	0.10		0	0.05

TABLE 2
UNIT TRANSMISSION COST ON EACH ARC IN EXAMPLE 2

Arc	Commodity	Cost	Arc	Commodity	Cost
a_1	1	2	a_4	1	1
	2	2		2	1
	3	3		3	2
a_2	1	2	a_5	1	2
	2	3		2	3
	3	3		3	3
a_3	1	1	a_6	1	2
	2	1		2	2
	3	2		3	3

Consider the network in Fig. 2. It is known that $U = (u_1, u_2, u_3, u_4, u_5, u_6) = (3, 2, 1, 1, 2, 3)$, $\mathbf{p} = (\rho_1, \rho_2, \rho_3) = (1, 2, 1)$, and there exists four MPs; $P^1 = \{a_1, a_2\}$, $P^2 = \{a_1, a_3, a_6\}$, $P^3 = \{a_2, a_4, a_5\}$, $P^4 = \{a_5, a_6\}$. Given $\mathbf{d} = (1, 1, 1)$ and $c = 16$, the family of $(\mathbf{d};\mathbf{c})$ -MPs is derived as follows:

Step 1. $L_1 = \min\{3,2\} = 2$, $L_2 = \min\{3,1,3\} = 1$, $L_3 = \min\{2,1,2\} = 1$, $L_4 = \min\{2,3\} = 2$,

$$W_1^1 = 2 + 2 = 4, W_1^2 = 2 + 3 = 5, W_1^3 = 3 + 3 = 6, W_2^1 = 2 + 1 + 2 = 5, W_2^2 = 2 + 1 + 2 = 5, W_2^3 = 3 + 2 + 3 = 8,$$

$$W_3^1 = 2 + 1 + 2 = 5, W_3^2 = 3 + 1 + 3 = 7, W_3^3 = 3 + 2 + 3 = 8, W_4^1 = 2 + 2 = 4, W_4^2 = 3 + 2 = 5,$$

$$\text{and } W_4^3 = 3 + 3 = 6.$$

Step 2. Find all feasible solutions $(f_1^1, f_1^2, f_1^3, f_2^1, f_2^2, f_2^3, f_3^1, f_3^2, f_3^3, f_4^1, f_4^2, f_4^3)$ subject to the following constraints by applying an implicit enumeration method:

$$\begin{cases} f_1^1 + f_2^1 + f_3^1 + f_4^1 = 1 \\ f_1^2 + f_2^2 + f_3^2 + f_4^2 = 1 \\ f_1^3 + f_2^3 + f_3^3 + f_4^3 = 1 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_2^1 \times 2 + f_3^1 \times 1 \leq 2 \\ f_1^2 \times 1 + f_2^2 \times 2 + f_3^2 \times 1 \leq 1 \\ f_1^3 \times 1 + f_2^3 \times 2 + f_3^3 \times 1 \leq 1 \\ f_4^1 \times 1 + f_4^2 \times 2 + f_4^3 \times 1 \leq 2 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_2^1 \times 1 + f_3^1 \times 2 + f_4^1 \times 2 + f_1^2 \times 1 + f_2^2 \times 1 \leq 3 \\ f_1^1 \times 1 + f_3^1 \times 1 + f_2^2 \times 2 + f_3^2 \times 2 + f_1^3 \times 1 + f_3^3 \times 1 \leq 2 \\ f_2^1 \times 1 + f_2^2 \times 2 + f_2^3 \times 1 \leq 1 \\ f_3^1 \times 1 + f_3^2 \times 2 + f_3^3 \times 1 \leq 1 \\ f_3^1 \times 1 + f_4^1 \times 1 + f_3^2 \times 2 + f_4^2 \times 2 + f_3^3 \times 1 + f_4^3 \times 1 \leq 2 \\ f_2^1 \times 1 + f_4^1 \times 1 + f_2^2 \times 2 + f_4^2 \times 2 + f_2^3 \times 1 + f_4^3 \times 1 \leq 3 \end{cases}$$

$$4f_1^1 + 5f_2^1 + 6f_3^1 + 5f_4^1 + 5f_2^2 + 8f_3^2 + 5f_3^3 + 7f_4^3 + 8f_3^3 + 4f_4^1 + 5f_4^2 + 6f_4^3 \leq 16$$

where f_j^ℓ is a nonnegative integer for $j = 1, 2, 3, 4$ and $\ell = 1, 2, 3$.

Total feasible solutions are $F^1 = (0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0)$, $F^2 = (0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1)$, $F^3 = (0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1)$, and $F^4 = (1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0)$.

Step 3. Transform such feasible solutions into (d;c)-MP candidates $X = (x_1, x_2, x_3, x_4, x_5, x_6)$ via

$$x_i = \sum_{\ell=1}^3 \sum_j \{f_j^\ell \rho^\ell \mid a_i \in P^j\} \quad \text{for } i = 1, 2, \dots, 6. \quad \text{Then } X^1 = (2, 1, 1, 0, 2, 3), \quad X^2 = (2, 2, 0, 0, 2, 2), \quad \text{and}$$

$$X^3 = (3, 2, 1, 0, 1, 2) \text{ are total (d;c)-MP candidates.}$$

Step 4. The network is cyclic, and $\{X^1, X^2, X^3\}$ is the family of all (d;c)-MP candidates. Since $X^i \not\prec X^j$, every (d;c)-MP candidate is a (d;c)-MP. The result is listed in Table 3.

TABLE 3
LIST OF ALL (D;C)-MPS IN EXAMPLE 2

(d;c)-MP candidate	(d;c)-MP?
$X^1 = (2, 1, 1, 0, 2, 3)$	Yes
$X^2 = (2, 2, 0, 0, 2, 2)$	Yes
$X^3 = (3, 2, 1, 0, 1, 2)$	Yes

VI. RELIABILITY EVALUATION

If $Y^1, Y^2, \dots, Y^{m_{(d;c)}}$ are the collection of all $(d;c)$ -MPs, then the system reliability for level $(d;c)$ is defined as $R_{(d;c)} = \Pr\{\bigcup_{i=1}^{m_{(d;c)}} \{X \mid X \geq Y^i\}\}$. To compute it, several methods such as inclusion-exclusion [8, 12], disjoint subset [13], and state-space decomposition [4] are available. Here we apply the state-space decomposition method to Example 2 and obtain that $R_{(d;c)} = \Pr\{\bigcup_{i=1}^{m_{(d;c)}} \{X \mid X \geq Y^i\}\} = 0.53235$ for demand level $d = (1,1,1)$ and $c = 16$.

VII. CONCLUSION

Given all MPs that are stipulated in advance, the proposed method can generate all $(d;c)$ -MPs of a multicommodity limited-flow network under budget constraints for each level $(d;c)$. The system reliability, i.e., the probability that k different types of commodity can be transmitted from the source node s to the sink node t in the way that the demand level $d = (d_1, d_2, \dots, d_k)$ is satisfied and the total transmission cost is less than or equal to c , can then be computed in terms of these $(d;c)$ -MPs. This algorithm can also apply to the limited-flow network with single commodity. Hence, earlier algorithm [18] is shown to be a special case of this new one.

REFERENCES

- [1] K. K. Aggarwal, J. S. Gupta, and K. G. Misra, "A simple method for reliability evaluation of a communication system," IEEE Transactions on Communication, Vol.23, 1975, pp. 563-565.
- [2] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Network Flows: Theory, Algorithms, and Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1993.
- [3] T. Aven, "Reliability evaluation of multistate systems with multistate components," IEEE Transactions on Reliability, Vol.34, 1985, pp. 473-479.
- [4] T. Aven, "Some considerations on reliability theory and its applications, Reliability Engineering and System Safety," Vol.21, 1988, pp. 215-223.
- [5] G. Bai, Z. Tian, and M. J. Zuo, "An improved algorithm for finding all minimal paths in a network," Reliability Engineering and System Safety, Vol.150, 2016, pp. 1-10.
- [6] S. G. Chen and Y. K. Lin, "Search for all minimal paths in a general large flow network," IEEE Transactions on Reliability, Vol.61, 2012, pp. 949-956.
- [7] P. Doulliez and J. Jamoulle, "Transportation networks with random arc capacities," RAIRO, Recherche Operationnelle, Vol.3, 1972, pp. 45-60.
- [8] E. El-Newehi, F. Proschan, and J. Sethuraman, "Multistate coherent systems," Journal of Applied Probability, Vol.15, 1978, pp. 675-688.
- [9] J. R. Evans, "Maximum flow in probabilistic graphs — the discrete case," Networks, Vol.6, 1976, pp. 61-183.
- [10] L. R. Ford and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, New Jersey, 1962.
- [11] E. Horowitz, S. Sahni, and S. Rajasekaran, Computer Algorithms/C++, Computer Science Press, New York, 1996.
- [12] J. C. Hudson and K. C. Kapur, "Reliability analysis for multistate systems with multistate components," IIE Transactions, Vol.15, 1983, pp. 127-135.
- [13] J. C. Hudson and K. C. Kapur, "Reliability bounds for multistate systems with multistate components," Operations Research, Vol.33, 1985, pp. 153-160.
- [14] C.C. Jane and Y. W. Lai, "A practical algorithm for computing multi-state two-terminal reliability," IEEE Transactions on Reliability, Vol.57, 2008, pp. 295-302.
- [15] C. C. Jane, Y. W. Lai, "A dynamic bounding algorithm for approximating multi-state two-terminal reliability," European Journal of Operations Research, Vol.205, 2010, pp. 625-637.
- [16] C. C. Jane and Y. W. Lai, "Computing multi-state two-terminal reliability through critical arc states that interrupt demand," IEEE Transactions on Reliability, Vol.59, 2010, pp. 338-345.
- [17] J. S. Lin, "Reliability evaluation of capacitated-flow network with budget constraints," IIE Transactions, Vol.30, 1998, pp.1175-1180.
- [18] J. S. Lin, C. C. Jane, and J. Yuan, "On reliability evaluation of a capacitated-flow network in terms of minimal pathsets," Networks, Vol.25, 1995, pp. 131-138.
- [19] Y. K. Lin, "Reliability evaluation for overall-terminal multistate flow networks with bi-directed arcs," Expert Systems with Applications, Vol.37, 2010, pp. 6669-6674.
- [20] Y. F. Niu and X. Z. Xu, "Reliability evaluation of multi-state systems under cost consideration," Applied Mathematical Modelling, Vol.36, 2012, pp. 4261-4270.
- [21] J. E. Ramirez-Marquez, D. W. Coit, and M. Tororella, "A generalized multistate-based path vector approach to multistate two-terminal reliability," IIE Transaction, Vol.38, 2006, pp. 477-488.

- [22] S. Satishan and K.C. Kapur, "An algorithm for lower reliability bounds of multistate two-terminal networks," IEEE Transactions on Reliability, Vol.55, 2006, pp. 199-206.
- [23] A. Shrestha, L. Xing, and D. W. Coit, "An efficient multistate multivalued decision diagram-based approach for multistate system sensitivity analysis," IEEE Transactions on Reliability, Vol.59, 2010, pp. 581-592.
- [24] W. C. Yeh, "Search for minimal paths in modified networks," Reliability Engineering and System Safety, Vol.75, 2007, pp. 389-395.
- [25] W. C. Yeh, "A simple universal generating function method to search for all minimal paths in networks," IEEE Transactions on System, Man and Cybernetics, Vol. 39, 2009, pp. 1247-1254.
- [26] W. C. Yeh, "Multistate network reliability evaluation under the maintenance cost constraint," International Journal of Production Economics, Vol.88, 2009, pp. 73-83.
- [27] W. C. Yeh, "An improved sum-of-disjoint-products technique for symbolic multistate flow network reliability, IEEE Transactions on Reliability, Vol.64, 2015, pp. 1185-1193.
- [28] M. J. Zuo, Z. Tian, and H.Z. Hung, "An efficient method for reliability evaluation of multistate networks given all minimal path vector," IIE Transaction, Vol.39, 2007, pp. 473-479.